BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

MATHEMATICS

On the Torsion-Free Abelian Groups with Hereditarily Generating Sequences

by
J. ŁOŚ

Presented by K. KURATOWSKI on December 1, 1955

The sequence $g_1, g_2, \ldots$ is called a hereditarily generating sequence (h. g. s.) of the Abelian group G, if each infinite subsequence $g_{k_1}, g_{k_2}, \ldots$ is a generating system of G.

The concept of an h. g. s. was introduced in [1]. We have also solved in that paper the structure problem for the torsion groups with h. g. s. and, moreover, we have proved that a group with h. g. s.

(I) *is a group of finite rank,*

(II) *has no finite and non-cyclic homomorphic images.*

The aim of this paper is to show, that conditions (I) and (II) are not only necessary but also sufficient for a torsion-free Abelian group to have an h. g. s.

From a result of B. H. Neumann (see [2], Theorem (4.4), p. 241) it follows that condition (II) is equivalent to the following one:

(II′) the group G is not a set-theoretical union of a finite number of its proper subgroups.

Therefore it is enough to show that conditions (I) and (II′) are sufficient for an Abelian torsion-free group to have an h. g. s.

1. A subgroup M of an Abelian group G is called maximal with respect to an element g in G, if $G/M = C_p{}^a$ *), where $a = 1, 2, \ldots$ or ∞, and $g + M$ is of order p.

With the help of Kuratowski-Zorn's Lemma it is easy to show that

1.1. If H is a subgroup of G and g is not in H, then there exists a subgroup M of G, maximal with respect to g with $H \subset M$.

In the group R^+ (additive group of rationals) there are only such maximal groups M, that $R^+/M = C_{p^\infty}$. Moreover, it is easy to see that

*) For $a < \infty$ the group $C_p{}^a$ is the cyclic group of order p^a, C_{p^∞} is the Prüffer group of the type p^∞ (p is assumed to be prime).

for each g in R^+ and for each prime number p there is exactly one subgroup M maximal with respect to g, such that $R^+/M=C_{p^\infty}$. In fact, as $g+M$ is of order p, pg is in M. But $R^+/\{pg\}=\sum_{i=1}^{\infty}C_{p_i^\infty}=S$, where $p_1,p_2,\ldots$ are all prime numbers and $\sum$ denotes the direct sum. In S there is only one subgroup A such that $S/A=C_{p^\infty}$ and $g,+\{pg\}\in A$, this is the direct sum of groups $C_{p_i^\infty}$ with $p_i\neq p$. Therefore M is the reciprocal image of A in the natural homomorphism of $R^+/\{pg\}$.

A subgroup M of G is called relatively maximal, if it is maximal with respect to some g in G. We have shown that there is in R^+ only a countable set of relatively maximal subgroups. This holds also for each torsion-free and divisible group G, i. e. for direct sums of a finite number of groups each of which is isomorphic to R^+: $G=R_1^+ +R_2^+ +\ldots+R_k^+$. In fact, each homomorphism $h(G)=C_{p^\infty}$ is determined by the homomorphisms $h(R_i^+)\subset C_{p^\infty}$, that is, either $h(R_i^+)=\{0\}$ or $h(R_i)=C_{p^\infty}$. For each $i\leqslant k$ there are only $\aleph_0$ such homomorphisms and for the group G there are $\aleph_0^k=\aleph_0$ such homomorphisms.

Let H be a subgroup of G and M a relatively maximal subgroup of H. It follows from (1.1) that M may be extended to a relatively maximal group M^* of G in such a way that the intersection of M^* and H is equal to M. If M and N are two different relatively maximal subgroups of H, then M^* and N^* are also different. Therefore if G has only a countable set of relatively maximal groups, then every subgroup H of G also has only a countable set of relatively maximal subgroups. As is known, each torsion-free Abelian group of finite rank can be imbedded in a torsion-free divisible group of finite rank.

Therefore we have the following

1.2. LEMMA. *Each torsion-free, Abelian group of finite rank has only a countable set of relatively maximal subgroups.*

2. THEOREM 1. *A group G which fulfills* (II′) *and has a countable set of relatively maximal subgroups only, has an h. g. s.*

Proof. Let $M_1, M_2,\ldots$ be all relatively maximal subgroups of G. For every n we can take an element g_n in G which is not in M_i for $i=1,2,\ldots,n$. Let $g_1',g_2',\ldots$ be a subsequence of the sequence $g_1,g_2,\ldots$ and let H be the group generated by the g_i'. Suppose that H is not equal to G, then H can be extended to a group M_n. But there exist only n elements g_i which can belong to M_n, therefore only n elements g_i' which can belong to M_n. Therefore we have $H=G$. This proves that $g_1,g_2,\ldots$ is an h. g. s. for G.

3. Theorem 1, lemma (1,2) and the mentioned results of [1] and [2] yield the following

THEOREM 2. *For a torsion-free Abelian group, conditions* (I) *and* (II) *are necessary and sufficient to have an h. g. s.*

The structure problem for mixed groups remains open.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] J. Łoś, E. Sąsiada and Z. Słomiński, *On groups with hereditarily generating systems*, Publ. Math. (to appear).

[2] B. H. Neumann, *Groups covered by permutable subsets*, J. London Math. Soc. **29** (1954), 236-248.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

MATHÉMATIQUE

Propriétés des méthodes continues de limitation du type de Borel

par

L. WŁODARSKI

Présenté par S. MAZUR à la séance du 20 Décembre 1955

Soit $x=\{\xi_n\}$ une suite de nombres complexes. Posons

$$(1)\qquad B_k(t,x)=2^{-k}e^{-t}\sum_{n=0}^{\infty}\frac{t^{n.2^{-k}}}{\Gamma(n\cdot 2^{-k}-1)}\xi_n,\qquad \text{où}\qquad t\geqslant 0\qquad \text{et}\qquad k=0,1,2,\ldots$$

$$(2)\qquad B(x)=\lim_{t\to\infty}B_k(t,x).$$

La suite $x=\{\xi_n\}$ sera dite *limitable par la méthode B_k vers le nombre $B_k(x)$* lorsque la série (1) est convergente pour $t\geqslant 0$ et que la limite (2) existe. L'expression (1) s'appellera *la transformée* de la suite x par rapport à la méthode B_k. Nous dirons que la transformée de la suite x par rapport à la méthode B_k existe lorsque la série (1) est convergente pour tout $t\geqslant 0$.

Chacune des méthodes B_k est une méthode continue de limitation [1] et assigne aux suites convergentes leurs limites habituelles. La méthode B_0 est la méthode classique de Borel. La fonctionnelle (2), définie par la méthode B_k dans un espace de suites qui contient des suites convergentes, est additive, c'est-à-dire que $B_k(x+y)=B_k(x)+B_k(y)$, et homogène, c'est-à-dire que $B_k(\lambda x)=\lambda B_k(x)$, les expressions $x+y$ et λx étant entendues au sens habituel.

La suite $x=\{\xi_n\}$ étant limitable par la méthode B_k vers le nombre ξ, celle qui en résulte en substituant une coordonnée ξ_n par une autre, $\bar{\xi}_n$, est également limitable par la méthode B_k vers le même nombre ξ et il en est le même de la suite $\{\xi_{n+1}\}$ formée en ajoutant une nouvelle coordonnée (cependant, en supprimant une coordonnée, la suite peut cesser d'être limitable par la même méthode; mais si elle continue de l'être, elle l'est vers le même nombre). La limitabilité de la suite $\{\xi_{n+1}\}$, à condition que la suite $\{\xi_n\}$ soit limitable, est appelée souvent *la translativité*

à droite de la méthode en question. On sait que toutes les méthodes additives, homogènes, qui conservent la limite habituelle et qui sont translatives à droite (ou à gauche), concordent entre elles dans une classe de suites divergentes. Si elles limitent par exemple la suite $0,1,0,1,0,1,\ldots$ c'est toujours vers le nombre $^1/_2$; de même, si elles limitent (vers un nombre fini) la progression géométrique (a^n) pour un $a \neq 1$, c'est toujours vers le nombre 0.

Nous allons signaler ici deux théorèmes sur la concordance des méthodes B_k. Ils s'appuient sur le lemme fondamental suivant:

LEMME. *Soit $f(u)$ une fonction complexe de variable réelle, définie pour $u \geqslant 0$ et continue. L'existence de la limite $\lim_{u\to\infty} f(u) = m$ entraîne celle de la limite $\lim_{t\to\infty} F(t) = m$, où*

$$F(t) = \frac{e^{-t}}{2\sqrt{\pi t}} \int_0^\infty e^{-u^2/4t + u} f(u)\, du.$$

THÉORÈME 1. *Lorsque la suite $x = \{\xi_n\}$ est limitable par la méthode B_p vers le nombre ξ et qu'il en existe la transformée par rapport à une méthode B_q, où $q \geqslant p$, cette suite est limitable aussi par la méthode B_q vers le même nombre ξ.*

Il s'ensuit de ce théorème que les méthodes B_k concordent entre elles.

On sait que la transformée de la suite $x = \{\xi_n\}$ par rapport à la méthode classique d'Abel-Poisson est donnée par l'expression

$$A(t,x) = (1-t)\sum_{n=0}^{\infty} t^n \xi_n \quad \text{où} \quad 0 \leqslant t < 1; \tag{3}$$

la suite x s'appelle limitable vers le nombre ξ par la méthode d'Abel lorsque

$$\lim_{t\to 1-} A(t,x) = \xi. \tag{4}$$

THÉORÈME 2. *Lorsque la suite $x = \{\xi_n\}$ est limitable par l'une des méthodes B_k vers le nombre ξ et qu'il en existe la transformée par rapport à la méthode d'Abel* (3), *cette suite est limitable aussi par la méthode d'Abel vers le même nombre ξ.*

Il s'ensuit de ce théorème que les méthodes B_k concordent avec la méthode classique d'Abel. Il s'ensuit également, en vertu des théorèmes connus sur la méthode d'Abel et sur celle de Borel B_0, que les méthodes B_k concordent également avec celles des moyennes, de Césaro et d'Euler.

Vu le théorème 1 sur la concordance entre les méthodes B_k, on peut en traiter la suite $\{B_k\}$ comme une seule méthode de limitation et qui est une généralisation naturelle de la méthode classique de Borel.

Convenons de dire que la suite x est *limitable par la méthode de Borel généralisée* lorsqu'elle l'est par l'une des méthodes B_k.

Nous appellerons domaine d'une méthode l'ensemble de suites limitables par celle-ci. Voici quelques théorèmes concernant les domaines des méthodes B_k.

Étant données deux méthodes B_p et B_q différentes ($p \neq q$), il existe toujours une suite limitable par l'une d'elles, mais pas par l'autre. Plus encore: il existe pour tout $k=0,1,2,\ldots$ une suite limitable par la méthode B_k, mais non limitable par aucune méthode B_p où $p>k$, de même qu'une suite limitable par la méthode B_k, mais non limitable par aucune méthode B_q où $q<k$.

La progression géométrique $\{a^n\}$ pour a complexes, à l'exception des nombres réels $a \geqslant 1$, est limitable par l'une des méthodes B_k vers le nombres 0. Pour $a=1$ elle l'est évidemment par chacune des méthodes B_k vers le nombre 1 et pour les $a>1$ réels elle l'est par chacune des méthodes B_k vers ∞, c'est-à-dire que $\lim_{t\to\infty} B_k(t,x)=\infty$.

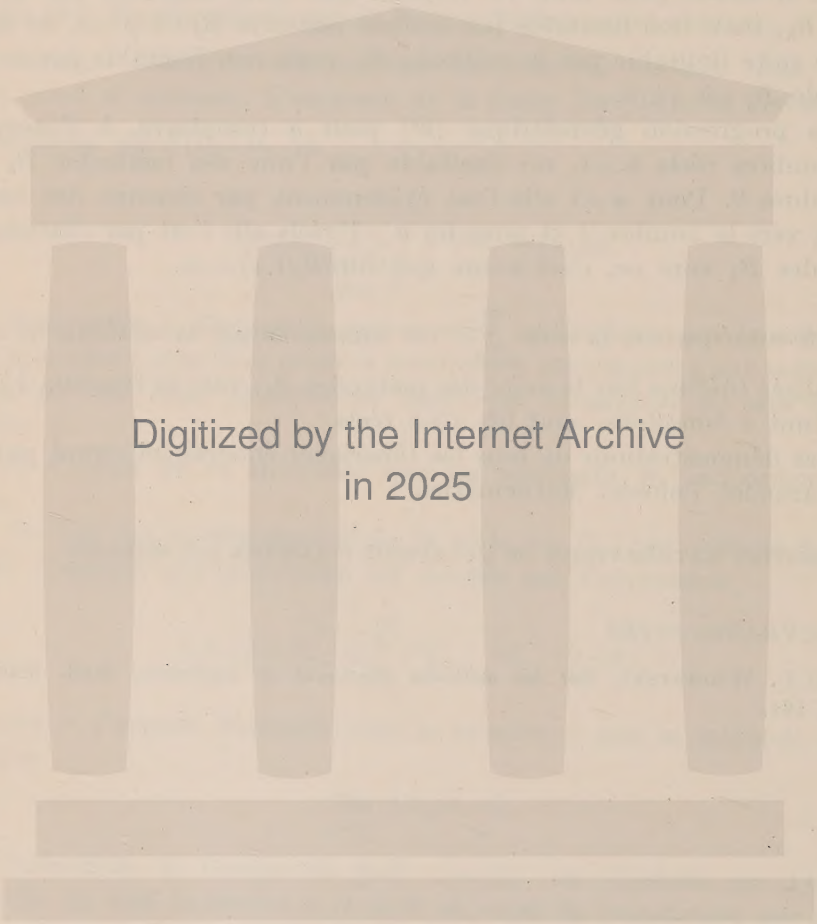
En conséquence, la série $\sum_{n=0}^{\infty} z^n$ est sommable par la méthode de Borel généralisée (définie par la suite des méthodes B_k) vers la fonction $1/(1-z)$ pour tout z complexe, sauf les $z \geqslant 1$ réels.

Les démonstrations de tous les théorèmes énoncés ici seront publiées dans Annales Polonici Mathematici.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

OUVRAGES CITÉS

[1] L. Włodarski, *Sur les méthodes continues de limitation*, Stud. Math. **14** (1954), 161.



Digitized by the Internet Archive
in 2025

https://archive.org/details/phytopathologia_1956_4_4

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

MATHÉMATIQUE

Sur la concordance entre les méthodes intégrales de sommation du type de Borel

par

L. WŁODARSKI

Présenté par S. MAZUR à la séance du 20 Décembre 1955

La méthode classique de Borel est bien connue comme continue et comme intégrale (p. ex. [3], p. 182-226, et [1], p. 120-146). Elle a trouvé des applications aux équations différentielles (p. ex. [1], p. 148-151), aux prolongements analytiques (p. ex. [1], chap. IV) et autres.

Une classe de méthodes intégrales du type de Borel a été considérée en 1900 par Le Roy [5], qui les a appliquées en suivant les idées de Borel, aux équations différentielles et aux prolongements analytiques. Ces méthodes définissent la somme généralisée d'une série $\sum_{n=0}^{\infty} \beta_n$ (β_n-complexes) par la formule

$$(1) \qquad \int_0^{\infty} e^{-t} \left[\sum_{n=0}^{\infty} \frac{t^{an}}{\Gamma(an+1)} \beta_n \right] dt, \quad \text{où} \quad a>0.$$

Perron a montré en 1920 ([4], p. 300, théorème 6) que la série $\beta_1+\beta_2+\ldots$ étant sommable par une telle méthode vers le nombre s, la série $\beta_0+\beta_1+\beta_2+\ldots$ l'est par la même méthode vers le nombre β_0+s.

Ces méthodes ont aussi été étudiées par Bromwich [2]. Ledit auteur a étudié une classe plus générale de méthodes, notamment celles appellées les méthodes des moments fixes ([3], p. 81-83).

Tous les auteurs cités démontraient la concordance partielle des méthodes en question, à savoir que toute série $\sum_{n=0}^{\infty} \beta_n$ sommable par deux de ces méthodes l'est par elles vers le même nombre, pourvu que les termes β_n satisfassent à des conditions supplémentaires convenables (p. ex. [3], p. 84-85).

On a les nouveaux théorèmes suivants sur la concordance de ces méthodes:

THÉORÈME 1. *Les méthodes définies par la formule* (1) *concordent (parfaitement) pour les* α *de la forme* 2^{-k}, *où* $k=0,1,2,\ldots$

THÉORÈME 2. *Toute méthode définie par la formule* (1) *pour* $\alpha=2^{-p}$, *où* $p=0,1,2,\ldots$, *concorde avec toute méthode continue de limitation du type borélien* B_k.

Pour la définition et les propriétés des méthodes continues de limitation du type de Borel, voir [6].

Une série $\sum_{n=0}^{\infty}\beta_n$ est dite sommable par la méthode d'Abel vers le nombre s, lorsque

1° il existe la transformée $\sum_{n=0}^{\infty}t_n-\beta_n$ pour $0\leqslant t<1$,

2° il existe la limite $\lim_{t\to 1-}\sum_{n=0}^{\infty}t_n\beta_n=s$.

THÉORÈME 3. *Lorsque la série* $\sum_{n=0}^{\infty}\beta_n$ *est sommable vers le nombre* s *par l'une quelconque des méthodes intégrales* (1) *pour* $\alpha=2^{-k}$, *où* $k=0,1,2,\ldots$, *et que sa transformée relative à la méthode d'Abel existe, cette série est sommable par la méthode d'Abel vers le même nombre* s.

Ce théorème entraîne la concordance entre les méthodes intégrales (1) pour $\alpha=2^{-k}$ et celle d'Abel.

On sait que toute série sommable par la méthode des moyennes ou par celle de Césaro l'est par la méthode d'Abel vers le même nombre. On sait aussi que toute série sommable par la méthode d'Euler l'est par celle de Borel vers le même nombre. Il s'ensuit en vertu des théorèmes 2 et 3 que les méthodes intégrales (1) pour α de la forme 2^{-k} concordent avec celles des moyennes, de Césaro et d'Euler.

Les démonstrations des théorèmes qui précèdent seront publiées dans Annales Polonici Mathematici.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

OUVRAGES CITÉS

[1] E. Borel, *Leçons sur les séries divergentes*, Paris, 1928.

[2] T. J. Bromwich, *An introduction to the theory of infinite series*, Cambridge, 1926.

[3] G. H. Hardy, *Divergent series*, Oxford, 1949.

[4] O. Perron, *Beitrag zur Theorie der divergenten Reihen*, Math. Zeitsch. **6** (1920), 286-310.

[5] É. Le Roy, *Sur les séries divergentes et les fonctions définies par un développement de Taylor*, Annales ... de l'Université de Toulouse ... (1900), 317-430.

[6] L. Włodarski, *Propriétés des méthodes continues de limitation du type de Borel*, ce numéro, p. 173.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

MATHEMATIK

Die Lösung im Grossen gemischter Probleme für Systeme inhomogener linearer Differentialgleichungen

von

L. MAURIN

Vorgelegt von W. RUBINOWICZ am 23. Dezember 1955

1. Einleitung

Die in dieser Arbeit angewandte Methode stützt sich auf die Spektraldarstellung der Funktionen eines selbstadjungierten Operators. Diese Methode ist auf Gleichungssysteme vom Typus

$$\frac{\partial^m u_i(x,t)}{\partial t^m} = (A_x u)_i(x,t) + f_i(x,t) \quad \text{worin} \quad m = 1,2;\ i = 1,2,...,r,\ x = (x_1,...x_n),$$

anwendbar, sofern der Operator A_x ein elliptisches, formal selbstadjungiertes System von Differentialoperatoren darstellt und sich zu einem selbstadjungierten Operator fortsetzen läßt.

Die wichtigsten Gleichungen in der Physik, u. a. die Diracsche Gleichung im Falle stationärer Felder, sind, wie es sich herausstellt, eben von diesem Typus.

Als Erster K. Yosida [1] verwendete systematisch die Methode der Funktionen der selbstadjungierten Fortsetzung eines symmetrischen Operators und erhielt auf diese Weise zum ersten Male den Beweis der Existenz im Großen für die Lösung des Cauchyschen Problems der Wellengleichung im Riemannschen Raume der Klasse C^∞.

Im Falle eines gemischten Problems kommt zu den Anfangsbedingungen noch eine homogene lineare Randbedingung hinzu. Diese Randbedingung läßt sich durch die Zugehörigkeit der Funktion zum Gebiet der selbstadjungierten Fortsetzung des Operators A_x formulieren, worauf K. Maurin [2] hingewiesen hat. (Er hat dabei die Resultate der Arbeit von M. I. Višik [3] ausgewertet, in der die eineindeutige Korrespondenz zwischen den homogenen Randbedingungen und der selbstadjungierten Fortsetzung des Operators nachgewiesen worden ist).

K. Maurin [2] hat gemischte Probleme für beliebige homogene Systeme vom Typus $\partial^m u_i(x,t)/\partial t^m = (Au)_i(x,t)$ mit $m = 1,2$ auf einer belie-

bigen, nicht notwendig kompakten differenzierbaren Mannigfaltigkeit gelöst. Die vorliegende Note bildet eben eine Fortsetzung dieser Arbeit. Hier wird die Lösung des gemischten Problems für, in ihren Anwendungen wichtige Systeme vom Typus $\partial^2 u_i/\partial t^2 = (A_x u)_i + f_i$, in der Form verallgemeinerter Fourierscher Integrale angegeben; außerdem werden die hinreichenden Existenzbedingungen der Lösung formuliert. Die obige Methode stellt daher eine theoretische Begründung der Fourierschen Methode dar.

Die Physiker verwendeten die Fouriersche Methode zur Lösung gemischter Probleme, ohne erst die Konvergenz der auftretenden Reihen zu prüfen. Im Falle einer hyperbolischen Gleichung zweiter Ordnung vom Typus

$$\frac{\partial^2 u}{\partial t^2} = A u + f(x,t) = \sum_{ik} \frac{\partial}{\partial x_i}\left[a_{ik}(x)\frac{\partial u}{\partial x_k}\right] + c(x)u + f(x,t)$$

(für einen negativ definiten Operator und eine Randbedingung vom Dirichlet-Neumannschen Typus in einem beschränkten Gebiete) hat O. A. Ladyschenskaja [4] die Konvergenz der Fourierschen Methode nachgewiesen. Die Methode von Ladyschenskaja ist aber nur auf Operatoren anwendbar, die ein ausschließlich diskretes Spektrum besitzen, was jedoch nur im Falle beschränkter Gebiete stattfindet. Ihre Untersuchungen erfordern überdies mühevolle Abschätzungen. Die Einfachheit und Allgemeinheit der in der vorliegenden Arbeit dargestellten Methode spricht dafür, daß eben sie den eigentlichen Weg zur Lösung des Problems darstellt. Im Spezialfalle des von Ladyschenskaja behandelten Problems besitzen die hier erhaltenen Bedingungen für die Anfangsfunktion, sowie für die Funktionen $f(x,t)$ eine andere Gestalt, als die von Ladyschenskaja angeführten. In der vorliegenden Note ist es gelungen, nachzuweisen, daß die hier geforderten Bedingungen für gewisse Dimensionszahlen, z. B. für die physikalisch wichtigen Fälle $n = 2,3$ schwächer sind, als die von Ladyschenskaja angegebenen. Der Vergleich zwischen den beiden Typen von Bedingungen gestaltete sich recht schwierig. Ladyschenskaja drückt nämlich ihre Bedingungen in der Terminologie des Sobolevschen Raumes $W_2^k(\Omega_n \times [O,T])$ aus, während sich unsere Bedingungen auf den Begriff des selbstadjungierten Operators und seiner Iteration stützen.

2. Bezeichnungen und Definitionen

Sei A ein System von r linearen Differentialoperatoren der σ-ten Ordnung von n unabhängigen Variablen $(x_1, x_2, \ldots, x_n) = x$, also

$$(1) \qquad (Au)(x) = (Au)_i(x) = \sum_{|\alpha| \leqslant \sigma} \sum_{j=1}^{r} a_{ij}^{\alpha}(x) D^{\alpha} u_j(x),$$

wo

$$u(x)=[u_1(x),u_2(x),\ldots,u_r(x)],\qquad \alpha=(\alpha_1\alpha_2,\ldots,\alpha_n),\qquad |\alpha|=\alpha_1+\alpha_2+\ldots+\alpha_n,$$

$$D^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots\partial x_n^{\alpha_n}};$$

$a_{ij}^\alpha(x)$ hingegen seien Funktionen, die in einem beliebigen offenen Bereiche Ω_n des n-dimensionalen Raumes komplexe Werte besitzen

$$a_{ij}^\alpha\in C^{\sigma+|\alpha|}(\Omega_n)\quad\text{für}\quad|\alpha|<\sigma\quad\text{und}\quad a_{ij}^\alpha\in C^{\max(|\alpha|+2,2\sigma)}\quad\text{für}\quad|\alpha|=\sigma$$

$$\text{wobei}\quad i,j=1,2,\ldots r.$$

Das System der Operatoren A heißt *elliptisch*, wenn die Matrix

$$A_{ij}(x,\xi)=\sum_{|\alpha|=\sigma}a_{ij}^\alpha(x)\,\xi^\alpha$$

nicht singulär ist für $x\in\Omega_n$ und für alle $\xi^\alpha=(\xi_1^{\alpha_1}\cdot\xi_2^{\alpha_2}\ldots\xi_n^{\alpha_n})$, deren Summe $\sum_{j=1}^{n}\xi_i^2\neq 0$ ist.

Das System der Operatoren A heißt formal selbstadjungiert, sobald

$$\sum_{|\alpha|\leqslant\sigma}\sum_{j=1}^{r}a_{ij}^\alpha(x)\,D^\alpha u_j(x)=\sum_{|\alpha|\leqslant\sigma}(-1)^{|\alpha|}D^\alpha\Big[\sum_{j=1}^{r}\overline{a_{ij}^\alpha(x)}\,u_j(x)\Big].$$

Ein *formal selbstadjungierter* Operator besitzt eine symmetrische Einschränkung; es ist dies nämlich die Einschränkung des Operators A, dessen Definitionsbereich die Menge der Systeme von r Funktionen ist, die σ-fach in Ω_n stetig differenzierbar sind und in Ω_n kompakte Träger besitzen. (Eine derartige Einschränkung des Operators A bezeichnet man mit A_0). Existiert eine selbstadjungierte Fortsetzung des Operators A_0, dann sagt man kurz, der Operator A lasse sich zu einem selbstadjungierten fortsetzen. Die selbstadjungierte Fortsetzung des Operators A_0 bezeichnet man mit $A_1=A_1^*\supset A_0$.

Der Operator A heißt *von unten halbbeschränkt*, wenn es eine solche reelle Zahl M gibt, daß $(A_0u,u)\geqslant M(u,u)$ für beliebige $u\in D(A_0)$, wobei $(u,v)\overset{\text{def}}{=}\int_{\Omega_n}\sum_{j=1}^{r}u^j(x)\overline{v^j(x)}\,dx$.

3. Die operatortheoretische Variante des gemischten Problems für Systeme:

$$\frac{\partial^2u_i}{\partial t^2}=-(Au)_i+f_i(x,t)$$

Im betrachteten klassischen Problem ist eine Lösung $u(x,t)=$ $=[u_1(x,t),\ldots,u_r(x,t)]$ des Systems

$$\frac{\partial^2u_i(x,t)}{\partial t^2}=-(Au)_i(x,t)+f_i(x,t)\qquad i=1,2,\ldots r \tag{2}$$

gesucht, (wo A ein Operator ist, dessen Eigenschaften in **2.** besprochen worden sind), das den Anfangsbedingungen

$$\begin{cases} u_i(x,0)=\varphi_i(x) \\ \dfrac{\partial u_i}{\partial t}(x,0)=\psi_i(x) \end{cases} \quad i=1,2\ldots r$$

genügt und die Randbedingung erfüllt, die sich (sobald $u(x,t)$ als Repräsentant eines einparametrigen Vektorsystems $u(-,t)\in L^{2,r}(\Omega)$ aufgefaßt wird) in der Form (a'): $u(-,t)\in D(A_1)$ anschreiben läßt. A_1 ist hierin eine gewisse selbstadjungierte Fortsetzung des Operators A. Eine derartige Lösung soll eine *klassische Lösung* heißen.

Setzt man in dem obigen Problem an Stelle der Systeme der Funktionen $u(x,t)$ und $f(x,t)=[f_1(x,t),\ldots,f_r(x,t)]$ Funktionen der Veränderlichen t mit Werten, die dem Raume $L^{2,r}(\Omega_n)$ entnommen sind und bezeichnet sie mit $u(-,t)$ bzw. $f(-,t)$; setzt man ferner an Stelle der Systeme der Funktionen $\varphi_i(x)$ und $\psi_i(x)$ die Vektoren φ und $\psi\in L^{2,r}(\Omega_n)$; weiters an Stelle der partiellen Ableitung $\partial^2/\partial t^2$ die starke Ableitung d^2/dt^2 und schließlich statt des Operators A die selbstadjungierte Fortsetzung A_1, dann erhält man die *operatortheoretische Variante des klassischen Problems*:

Gesucht ist die Lösung $u(-,t)\in L^{2,r}(\Omega_n)$ der Gleichung

$$(2') \qquad \frac{d^2u(-,t)}{dt^2}=-A_1u(-,t)+f(-,t)$$

die den Anfangsbedingungen

$$(a) \qquad \begin{aligned} u(-,0)&=\varphi \\ \frac{du(-,0)}{dt}&=\psi \end{aligned}$$

genügt. (Die Bedingung $u(-,t)\in D(A_1)$ kommt schon in der Gestalt der Gleichung $(2')$ zum Ausdruck).

Die Lösung der operatortheoretischen Variante des gegebenen Problems wird im folgenden als verallgemeinerte Lösung bezeichnet werden. Es ist nämlich bewiesen worden, daß die Lösung des klassischen Problems im Falle eines beschränkten Gebietes Ω_n die Lösung seiner operatortheoretischen Variante (exakt ausgedrückt: der Repräsentant der Lösung der operatortheoretischen Variante $u(-,t)$) ist.

Gestützt auf die Ergebnisse von K. Maurin [2] für homogene Systeme und unter Berücksichtigung der Resultate der Arbeit von L. Gårding [5], (L. Gårding drückt hierin die Funktion des selbstadjungierten elliptischen Operators durch das verallgemeinerte Fouriersche Integral aus) wurde bewiesen:

SATZ 1. *Ist A ein System von Operatoren der Form* (1): *elliptisch, formal selbstadjungiert und von unten halbbeschränkt; ist ferner A_1 eine beliebige selbstadjungierte Fortsetzung von A, wobei $\varphi \in D(A_1)$, $\psi \in D(A_1^{1/2})$, $f(-,\tau) \in D(A_1^{1/2})$ für jedes $\tau \geqslant 0$ und für fast alle τ gleich einer hinsichtlich τ stark stetigen Funktion; ist schliesslich $A_1^{1/2} f(-,\tau)$ eine Funktion mit Werten aus $L^{2,r}(\Omega_n)$, die im Bochnerschen Sinne* [6] *im Intervalle $[0,T]$ integrierbar ist, dann besitzt dieses gemischte Problem für das System*

$$\frac{\partial^2 u_i}{\partial t^2}(x,t) = -(Au)_i(x,t) + f_i(x,t)$$

eine verallgemeinerte Lösung in der Gestalt

$$(3) \qquad u(x,t) = \operatorname*{l.i.m}_{L^{2,r}(\Omega_n)} \int_M^{\infty} \sum_{k=1}^{N(\lambda)} \cos(\lambda^{1/2} t (F\varphi)_k(\lambda) e_k(\lambda,x)\, d\mu(\lambda) +$$

$$+ \operatorname*{l.i.m}_{L^{2,r}(\Omega_n)} \int_M^{\infty} \sum_{k=1}^{N(\lambda)} \lambda^{-1/2} \sin(\lambda^{1/2} t)(F\psi)_k(\lambda) e_k(\lambda,x)\, d\mu(\lambda) +$$

$$+ \int_0^t \Big\{ \operatorname*{l.i.m}_{L^{2,r}(\Omega_n)} \int_M^{\infty} \sum_{k=1}^{N(\lambda)} \lambda^{-1/2} \sin[\lambda^{1/2}(t-\tau)](Ff)_k(\lambda) e_k(\lambda,x)\, d\mu(\lambda) \Big\} d\tau$$

$$\text{worin} \qquad (F\varphi)_k(\lambda) = \operatorname*{l.i.m}_{L^2(\mu)} \int_{\Omega_n} \overline{e_k(\lambda,x)}\, \varphi(x)\, dx,$$

wobei $e_k(\lambda,x)$ sind Eigenfunktionen des Operators A die dem Eigenwerte λ entsprechen; μ ist ein positives Mass auf der Achse (M, ∞).

Gestützt auf die Eindeutigkeit der Lösung der Gleichung (2′) unter den Bedingungen (a) bewiesen wir folgendes

SATZ 2. *Sei $u_1(-,t)$ die Lösung der Gleichung $\dfrac{d^2 u(-,t)}{dt^2} = -A_1 u(-,t) + f_1(-,t)$ mit den Anfangsbedingungen $u(-,0) = \varphi_1$ $\dfrac{du}{dt}(-,0) = \psi_1$; $u_2(-,t)$ hingegen sei die Lösung der Gleichung: $d^2 u(-,t)/dt^2 = -A_1 u(-,t) + f_2(-,t)$ mit den Bedingungen: $u(-,0) = \varphi_2$, $\dfrac{du}{dt}(-,0) = \psi_2$. Es gibt eine solche Zahl $M(T)$, dass für jedes $t \in [0,T]$, $\|u_1(-,t) - u_2(-,t)\| < M(T)\big(\|\varphi_1 - \varphi_2\| + \|\psi_1 - \psi_2\| + \sup_{t\in[0,T]} \|f_1(-,t) - f_2(-,t)\|\big)$ (Korrektheit des Problems!).*

4. Klassische Lösung des gemischten Problems für das System:

$$\frac{\partial^2 u_i}{\partial t^2} = -(Au)_i + f_i(x,t)$$

Das Problem der Existenz der klassischen Lösung läßt sich auf die Frage zurückführen, unter welchen Bedingungen ein Repräsentant der allgemeinen Lösung existiert, der stetige Ableitungen der σ-ten Ordnung

nach x und eine zweite Ableitung nach t besitzt. Die Beantwortung dieser Frage bildete die wesentliche Schwierigkeit der dargestellten Methode. Das Problem wurde in Anlehnung an die von K. Maurin, im Falle homogener Systeme, angewandte Methode gelöst. Die genannte Methode stützt sich auf einen von K. Maurin bewiesenen Fundamentalsatz über schwache Lösungen elliptischer Systeme [7]. Es wurde der Beweis erbracht, daß der

Satz 3. *gilt: Die verallgemeinerte Lösung* (3) *ist eine klassische Lösung, sobald für* $m=[n/2\sigma]+2$, $\varphi \in D(A_1^m)$, $\psi \in (A_1^{m-1/2})$, $f(-,\tau) \in D(A_1^{m-1/2})$ *für jedes* $\tau \in [0,T]$, *wobei* $A_1^{m-1/2} f(-,\tau)$ *eine im Bochnerschen Sinne integrierbare Funktion mit Werten aus* $L^{2,r}(\Omega_n)$ *ist, während* $A_1^{m-1} f(-,\tau)$ *eine Funktion darstellt, deren Werte aus* $L^{2,r}(\Omega_n)$ *entnommen sind und die für fast alle* τ *einer in Bezug auf* τ *stark stetigen Funktion gleich ist.*

Da das Maß $\mu(\lambda)$ sich auf das Spektrum des Operators A_1 konzentriert, reduzieren sich im Falle eines ausschließlich diskreten Spektrums die Integrale im Ausdrucke (3) auf die entsprechenden Fourierreihen.

Der Vergleich der hinreichenden Bedingungen für die Existenz einer klassischen Lösung (die auf Grund des Theorems 3. erhalten wurde) in dem von Ladyschenskaja betrachteten Falle, d. h. im Falle einer Gleichung zweiter Ordnung

$$\frac{\partial^2 u}{\partial t^2} = \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + a_0 u + f(x,t) = Au + f(x,t)$$

bei positiv definitem Operator A und beschränktem Gebiete Ω_n mit den von Ladyschenskaja angeführten Bedingungen, ergab folgenden

Satz 4. *Die in der vorliegenden Arbeit geforderten Bedingungen sind für die Dimensionszahlen* $n=4l+2$ *und* $n=4l+3$, *wo* $l=0,1,2\ldots$, *also für die physikalisch besonders wichtigen Zahlen* 2 *und* 3 *schwächer, als die von Ladyschenskaja angegebenen.*

Wie schon in der Einleitung bemerkt worden ist, bietet der Vergleich zwischen den hier angeführten und den von Ladyschenskaja geforderten Bedingungen große Schwierigkeiten, mit Rücksicht darauf, daß die Bedingungen von Ladyschenskaja in der Bezeichnungsweise des Sobolevschen Raumes $W_2^k(\Omega_n)$, bzw. $W_2^k(\Omega_n \times [0,T])$ angegeben sind. Der Vergleich der Bedingungen in den beiden Darstellungstypen forderte infolgedessen eine Untersuchung der Integrierbarkeit der Funktion $\in W_2^k(\Omega_n \times [0,T])$ im Bochnerschen Sinne, sowie eine Untersuchung des Zusammenhanges zwischen der selbstadjungierten Fortsetzung eines Operators und dem Sobolevschen Differentiationsoperator — eines Zusammenhanges, der sich nur teilweise herstellen läßt.

5. Die Diracschen Gleichungen im Falle stationärer Felder

Die Diracsche Gleichung kann man für den Fall stationärer Felder in folgender Form aufschreiben:

$$\frac{\hbar}{i}\frac{\partial u}{\partial t} = -Hu, \tag{4}$$

wo

$$H = -i\sum_{k=1}^{3}\alpha^k\left[\frac{\partial}{\partial x_k} + \frac{ie}{c}\Phi_k(x)\right] - \frac{e}{c}\Phi_0(x) + mc\beta \equiv -A + mc\beta - \frac{e}{c}\Phi_0(x),$$

$\Phi_k \in L^2(\Omega)$, $|\Phi_0(x)| \leqslant M$ und α_k, β Diracsche Matrizen sind.

Der Operator H ist elliptisch, weil

$$\det\left(\sum_{k=1}^{3}\alpha^k\xi_k\right) = -(\xi_3^2 + \xi_1^2 + \xi_2^2)^2 \neq 0$$

Mit Rücksicht darauf, daß die Matrizen α^k und β hermetisch sind und im Hinblick darauf, daß $\Phi_k(x)$ und $\Phi_0(x)$ Funktionen darstellen, deren Werte reell sind, ist es ein Leichtes nachzuweisen, daß die Einschränkungen H_0 und A_0 der Operatoren H und A symmetrisch sind.

Gestützt auf ein Theorem von v. Neumann *ist der Beweis erbracht worden, dass sich der Operator A_0 auf einen selbstadjungierten fortsetzen lässt (er hat die gleichen Defektindizes!). Aus der Beschränktheit des Operators $mc\beta - e\Phi_0(x)/c$ folgt, dass der Operator $H_0 = -A_1 + mc\beta - e\Phi_0/c$ sich auf einen selbst adjungierten fortsetzen lässt.*

SCHRIFTTUM

[1] K. Yosida, *On Cauchy's problem in the large for wave equations*, Proc. Japan Acad., **28** (1952), 396.

[2] K. Maurin, *Über gemischte Rand- und Anfangswertprobleme im Großen für eine Klasse von Gleichungssystemen auf differenzierbaren Mannigfaltigkeiten. Eine Begründung der Fouriermethode.* Bull. Acad. Polon. Sci., Cl. III, **3** (1955), 471.

[3] M. J. Višik, *Über allgemeine Randwertprobleme für elliptische Differenzialgleichungen*, (auf Russisch), Trudy Mosk. Matem. Obschtestva **1** (1952), 187.

[4] O. A. Ladyschenskaja, *Das gemischte Problem für die hyperbolische Gleichung*, (auf Russisch), Moskau (1953).

[5] L. Gårding, *Applications of the theory of direct integrals of Hilbert spaces to some integral and differential operators*, University of Maryland, The Institute for Fluid Dynamics and Applied Mathematics, Lecture Series No. 11 (1954).

[6] E. Hille, *Functional analysis and semigroups*, New York 1948, p. 63.

[7] K. Maurin, *Der Fundamentalsatz über schwache Lösungen der allgemeinen linearen Systeme der elliptischen Differentialgleichungen beliebiger Ordnung*, Bull. Acad. Polon. Sci., Cl. III, **2** (1954), 457.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

MATHEMATICS

Theorem on Antipodes for Multi-Valued Mappings and a Fixed Point Theorem

by

J. W. JAWOROWSKI

Presented by K. BORSUK on December 24, 1955

1. Introduction. Some theorems concerning single-valued functions remain true for certain multi-valued mappings. For example, there exists such a generalization of the fixed point theorems [2]. The Vietoris Mapping Theorem (see below) plays an important role in such generalizations. The purpose of this paper is to give a generalization of Borsuk's theorem on antipodes ([1], p. 178) for the case of mappings of the sphere with values which are acyclic subsets of Euclidean space.

2. The sphere S_n. Let S_n be the unit sphere in the $(n+1)$-dimensional Euclidean space E_{n+1}, i. e., $S_n = \underset{x \in E_{n+1}}{E} [|x|=1]$. The antipodal mapping of S_n will be denoted by a, i. e., $a(x) = -x$. Thus a is a continuous involution of S_n.

3. Notions of homology. Let X be a metric space. We shall use the Vietoris homology theory modulo 2 of X, and we shall denote by $H^i(X)$ the i-dimensional homology group modulo 2 of X. The space X is called *n-acyclic*, if $H^i(X)=0$ for every $-1 \leqslant i < n$; X is called *acyclic*, if $H^i(X)=0$ for every i. Let us point out that by the boundary of a 0-dimensional simplex consisting of a single point of X, we understand the rest 1 modulo 2, and we consider the rests 0 and 1 modulo 2 as (-1)-dimensional cycles. Consequently, X is 0-acyclic if, and only if, it is not empty. The sphere S_n is n-acyclic but not $(n+1)$-acyclic

Given a continuous mapping $f: X \to Y$ of a compact space X into another such space Y, we shall denote by $\bar{f}_i$ the homomorphism $\bar{f}_i: H^i(X) \to H^i(Y)$ induced by f. We shall use the following

VIETORIS MAPPING THEOREM [6]. *Let $f: X \to Y$ be a continuous mapping of a compact space X into Y, such that $f^{-1}(y)$ is n-acyclic for every $y \in Y$. Then the homomorphism $\bar{f}_i$ induced by f is an isomorphism $\bar{f}_i: H^i(X) \approx H^i(Y)$, for every $-1 \leqslant i < n$.*

4. Multi-valued functions. Let X and Y be compact metric spaces and let $F: X \to Y$ be a multi-valued function which assigns to every point $x \in X$ a non empty subset $F(x)$ of Y. The subset $Z = \underset{(x,y)}{E}[y \in F(x)]$ of the Cartesian product $X \times Y$ will be called the *graph* of F. The function F is said to be *continuous* if the graph of F is closed in $X \times Y$. The continuity of F implies that the sets $F(x)$ are compact, and it is equivalent to the upper semicontinuity of the function F regarded as a single valued mapping of X into the hyperspace 2^Y of non-empty compact subsets of Y. Obviously, if $F: X \to Y$ is single-valued, i. e., if every set $F(x)$ consists of a single point $f(x)$, then the continuity of F means the ordinary continuity of the singled-valued function f.

The continuous multi-valued function $F: X \to Y$ is said to be *n-acyclic* (or *acyclic*) if the set $F(x)$ is n-acyclic (or acyclic) for every $x \in X$.

Let the multi-valued function $F: X \to Y$ be continuous and n-acyclic. Then the graph Z of F is compact. Let us consider the projections of Z

$$f: Z \to X, \qquad g: Z \to Y,$$

defined by $f(x,y) = x$, $g(x,y) = y$. Then f and g are continuous and the sets $f^{-1}(x)$ are homeomorphic to $F(x)$. Consequently, $f^{-1}(x)$ is n-acyclic for every $x \in X$, and by Vietoris's theorem, f induces an isomorphism $\bar{f}_i: H^i(Z) \approx H^i(X)$ for every $i < n$. Hence the homomorphism

$$\bar{F}_i = \bar{g}_i \bar{f}_i^{-1}: H^i(X) \to H^i(Y)$$

is defined for every $i < n$. The homomorphism $\bar{F}_i$ will be said to be *induced* by the multi-valued function F.

The graph of the identity in X, i. e., the subset $D = \underset{(x,y)}{E}[x = y]$ of $X \times X$ is the diagonal of $X \times X$. A multi-valued function $\Phi: X \to X$ will be called a *multi-valued involution* if the graph of Φ is symmetric with respect to the diagonal, i. e., if $y \in \Phi(x)$ implies $x \in \Phi(y)$. This notion is a generalization of the notion of a singled-valued involution.

5. Theorems on antipodes.

THEOREM 1. *Let M be a compact n-acyclic space and let $\Phi: M \to M$ be a multi-valued continuous n-acyclic involution of M. Let $F: M \to S_n$ be a multi-valued continuous acyclic mapping of M into S_n such that, for every $(x,y) \in M \times M$ with $y \in \Phi(x)$, the sets $F(x)$ and $F(y)$ are disjoint. Then the homomorphism $\bar{F}_n: H^n(M) \to H^n(S_n)$ induced by F is a homomorphism onto.*

Proof. Let us consider the graph of F

$$X = \underset{(x,u)}{E}[u \in F(x)] \subset M \times S_n.$$

Since F is continuous, then X is compact. The projections $f: X \to M$, $g: X \to S_n$, defined by

(1) $$f(x,u) = x, \qquad g(x,u) = u$$

are continuous. Since F is n-acyclic, then by **4**, f induces an isomorphism

$$\bar{f}_i : H^i(X) \approx H^i(M), \quad \text{for every} \quad i, \tag{2}$$

and

$$\bar{F}_n = \bar{g}_n \bar{f}_n^{-1}. \tag{3}$$

Let

$$Y = \underset{(x,y,u)}{E}\,[y \in \Phi(x);\ u \in F(x)] \subset M \times M \times S_n.$$

Thus, Y is compact and the projection $p : Y \to X$ defined by

$$p(x,y,u) = (x,u) \tag{4}$$

is continuous. Since the set $p^{-1}(x,u) = \underset{(x,y,u)}{E}\,[y \in \Phi(x)]$ is homeomorphic to $\Phi(x)$, and since Φ is n-acyclic, it follows that $p^{-1}(x,u)$ is n-acyclic for every $(x,u) \in X$. Hence, p induces an isomorphism

$$\bar{p}_i : H^i(Y) \approx H^i(X), \quad \text{for every} \quad i < n. \tag{5}$$

Let

$$Z = \underset{(x,y,u,v)}{E}\,[y \in \Phi(x);\ u \in F(x);\ v \in F(y)] \subset M \times M \times S_n \times S_n.$$

Thus Z is compact, the projection $Q : Z \to Y$ defined by

$$q(x,y,u,v) = (x,y,u) \tag{6}$$

is continuous and, for every $(x,y,u) \in Y$, the set $q^{-1}(x,y,u) = \underset{(x,y,u,v)}{E}\,[v \in F(y)]$ is homeomorphic to $F(y)$. Consequently, it is acyclic, and therefore q induces an isomorphism

$$\bar{q}_i : H^i(Z) \approx H^i(Y), \quad \text{for every} \quad i. \tag{7}$$

From (5) and (7) we conclude that the mapping $pq : Z \to X$ induces the isomorphism

$$\bar{p}_i \bar{q}_i : H^i(Z) \approx H^i(X), \quad \text{for every} \quad i < n. \tag{8}$$

By (2) and (8) the mapping $fpq : Z \to M$ induces the isomorphism

$$\bar{f}_i \bar{p}_i \bar{q}_i : H^i(Z) \approx H^i(M), \quad \text{for every} \quad i < n.$$

Since M is n-acyclic, it follows that Z is also n-acyclic.

Let us observe that, if $(x,y,u,v) \in Z$, then $(y,x,v,u) \in Z$. Hence the mapping $\psi : Z \to Z$ defined by

$$\psi(x,y,u,v) = (y,x,v,u) \tag{9}$$

is a continuous (single-valued) involution of Z.

Let $h: Z \to S_n$ be the projection of Z onto S_n defined by

$$h = gpq. \tag{10}$$

Then, by (6), (4) and (1), $h(x,y,u,v) = u$.

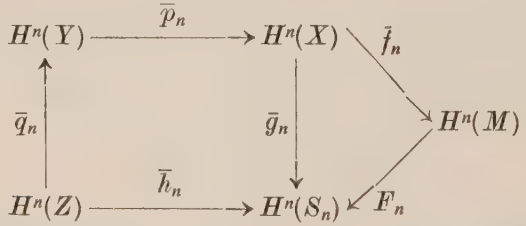
Moreover,

$$h(z) \neq h\psi(z), \quad \text{for every} \quad z \in Z. \tag{11}$$

Indeed, if we suppose that, for some $z = (x,y,u,v) \in Z$, $h(z) = h\psi(z)$, then by (9), $u = v$. By the definition of Z, this means that $y \in \Phi(x)$, $u \in F(x)$, $u \in F(y)$. Consequently, the sets $F(x)$ and $F(y)$ have the common point u. But this contradicts the hypothesis that $F(x)$ and $F(y)$ are disjoint for $y \in \Phi(x)$.

Hence ψ is a continuous involution of the acyclic compact space Z, and h is a continuous mapping of Z onto S_n, satisfying condition (11). Hence by [3], (Theorem 1, p. 290), we conclude that the induced homomorphism $\bar{h}_n: H^n(Z) \to H^n(S_n)$ is onto.

Let us consider the diagram



according to relations (3) and (10). Since $\bar{h}_n$ is onto, then $\bar{g}_n$ is onto. It follows that $\bar{F}_n$ is onto, and Theorem 1 is proved.

Corollary 1. *Let M be an n-acyclic compact space and let Φ be a multi-valued continuous n-acyclic involution of M. Let F be a multi-valued continuous n-acyclic mapping of M into S_n such that, for every $(x,y) \in M \times M$ with $y \in \Phi(x)$, the sets $F(x)$ and $F(y)$ are disjoint. Then F maps M onto S_n, i. e.,* $\sum_{x \in M} F(x) = S_n$.

Proof. We can assume that F is acyclic, since the only n-acyclic but not acyclic subset of S_n is the whole S_n. Suppose that F maps M into a proper subset of S_n. Then the induced homomorphism $\bar{F}_n: H^n(M) \to H^n(S_n)$ maps $H^n(M)$ into zero. Since the group $H^n(S_n)$ is not zero, we obtain a contradiction with Theorem 1, which proves Corollary 1.

Theorem 2. *Let M be an n-acyclic compact space and let Φ be a multi-valued continuous n-acyclic involution of M. Let F be an upper semi-continuous function which assigns to every $x \in M$ a compact acyclic subset of E_n. Then there exists a pair $\{x_0, y_0\}$ such that $y_0 \in \Phi(x_0)$ and $F(x_0) \cdot F(y_0) \neq 0$.*

For, the function F may be regarded as a multi-valued, continuous, acyclic mapping of M into a proper subset of S_n. If we suppose that $F(x)\cdot F(y)=0$, for every pair $\{x,y\}$ such that $y\in\Phi(x)$, then we obtain a contradiction of Corollary 1.

In a special case, when $M=S_n$ and Φ is a single-valued antipodal mapping a, we have

COROLLARY 2. *Let F be an upper semi-continuous mapping which assigns to every point $x\in S_n$ a compact acyclic subset $F(x)$ of E_n. Then there exists a point $x_0\in S_n$, such that $F(x_0)\cdot F(a(x_0))\neq 0$.*

Theorems 1 and 2 are generalizations of Borsuk's Theorems I and II from [1] (p. 178), respectively.

Remark. The assumptions of Theorems 1 and 2 that the functions Φ and F are acyclic is essential. For example, let a be a point of S_1, and let F be the multi-valued function which takes a onto two end-points of the segment I: $-1\leqslant t\leqslant 1$ and which maps homeomorphically the set $S_1-(a)$ onto the open segment $-1<t<1$. Thus F is 1-acyclic, the function $F: S_1\to E_1$ is continuous and $F(x)\cdot F(a(x))=0$, for every $x\in S_1$. But $F(a)$ is not 1-acyclic.

For the second example, let Φ be the multi-valued involution of the segment I defined by: $\Phi(t)=t+1$, for $-1\leqslant t<0$, $\Phi(0)=(-1)+(1)$ $\Phi(t)=t-1$, for $0<t\leqslant 1$. Then Φ is continuous, I is 1-acyclic (even acyclic), but $t\notin\Phi(t)$, for every $t\in I$. The identity mapping i on I maps I into E_1 such that $i(x)\neq i(y)$, for every $y\in\Phi(x)$. However, $\Phi(0)$ is not 1-acyclic.

6. Fixed points of multi-valued involutions. Let F be a multi-valued mapping of M into itself. A point $x\in M$ is said to be a fixed point of F, if $x\in F(x)$. The theorem of S. Eilenberg and D. Montgomery [2] implies that every multi-valued continuous acyclic mapping of an acyclic compact ANR-set M has a fixed point. However, when F is an involution, the assumption that M is an ANR-set can be omitted:

THEOREM 3. *Let Φ be a multi-valued continuous acyclic involution of a compact acyclic space M. Then Φ has a fixed point.*

Proof. The graph $X=\underset{(x,y)}{E}[y\in\Phi(x)]$ of Φ is compact. It is acyclic, since Φ and M are acyclic (see **4**). The mapping $\psi: X\to X$ defined by $\psi(x,y)=(y,x)$ is a continuous (single-valued) involution of X. It follows that ψ has a fixed point (see [4] or [3], Theorem 4, p. 291). Hence there exists a point $x_0\in M$, such that $x_0\in\Phi(x_0)$.

The second example of the Remark shows that the hypothesis of the acyclicity of Φ is essential.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. **20** (1933), 177-190.

[2] S. Eilenberg and D. Montgomery, *Fixed point theorems for multi-valued transformation*, Amer. Journ. of Math. **58** (1946), 214-222.

[3] J. W. Jaworowski, *Involutions of compact spaces and a generalization of Borsuk's theorem on antipodes*, Bull. Acad. Polon. Sci., Cl. III, **3** (1955), 289-292.

[4] S. Lefschetz, *Algebraic Topology*, (App. B: P. A. Smith, *Fixed points of periodic transformations*).Coll. Pub. Amer. Math. Soc., **27** (1942), 350-373.

[5] L. Vietoris, *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen*, Math. Ann. **97** (1927), 454-472.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

MATHÉMATIQUE

Sur la fixation des décompositions

par

B. KNASTER

Présenté par K. BORSUK le 18 Janvier 1956

Soit f une fonction continue, définie dans un espace métrique compact X et dont Y est l'ensemble des valeurs. La décomposition de X en ensembles compacts disjoints — dits *tranches* —

$$X=\sum_{y\in Y} f^{-1}(y) \tag{1}$$

est *semi-continue* ([1], p. 42); elle est *continue* lorsque la transformation f de X en Y est *intérieure* ([1], p. 47); enfin, elle est *monotone* — et la fonction f l'est dite aussi ([1], p. 123) — lorsque les tranches $f^{-1}(y)$ sont des continus.

Kosiński [2] a démontré que la décomposition (1) de X étant continue et monotone, la dimension de Y étant finie et le diamètre de tout $f^{-1}(y)$ où $y \in Y$ dépassant une constante positive donnée, il existe dans X, pour tout $\varepsilon>0$, une somme finie $G=G_1+\ldots+G_{k(\varepsilon)}$ d'ensembles ouverts dont les fermetures sont disjointes, dont les diamètres sont inférieurs à ε et qui *fixent* la décomposition (1), c'est-à-dire que G empiète sur chaque tranche

$$G\cdot f^{-1}(y)\neq 0 \quad \text{pour tout} \quad y\in Y. \tag{2}$$

La présente communication a pour but de démontrer que, toutes les hypothèses gardées, celle de *continuité de la décomposition* (1) est essentielle pour l'existence d'une telle fixation (voir [2], p. 72, Problem 2). Les exemples I et II qui suivent montrent que la présence d'une seule tranche de discontinuité, bien qu'encore de cohésion ([1], p. 47 et 140), peut rendre impossible la fixation de la décomposition (1) pour des $\varepsilon>0$ suffisamment petits.

Dans l'exemple I, Y est la suite harmonique avec sa limite 0, donc un ensemble dénombrable, et dans l'exemple II, qui contient le précédent, Y est le segment $0\leqslant y\leqslant 1$, donc un continu, et il en est de même

de X. Dans les deux exemples, $f^{-1}(0)$ est la seule tranche de discontinuité et la fixation est en défaut déjà pour $\varepsilon=1/2$.

Désignons par I le segment $0\leqslant x\leqslant 1$, par C le carré $I\times I$ dans le plan $y=0$, par $\{Q_n\}$ — la suite de tous les cercles ouverts du même plan aux longueurs des rayons et aux coordonnées des centres rationnelles — appelons-les *cercles rationnels* — et qui empiètent sur C, par $\{S_n\}$ — la suite de toutes les sommes finies de termes de la suite $\{Q_n\}$, chaque somme étant de diamètre inférieur à 1:

$$S_n=\sum_{i=1}^{j_n} Q_{n_i} \quad \text{où} \quad \delta(S_n)<1 \quad \text{pour} \quad n=1,2,\ldots, \tag{3}$$

et par $\{U_n\}$ — la suite de tous les ensembles ouverts (du même plan) tels que

$$U_n=\sum_{j=1}^{k_n}{}^{*} S_{n_j} \quad \text{où} \quad \bar{S}_{n_j}\cdot\bar{S}_{n_{j'}}=0 \tag{4}$$

pour $j\neq j'$ parcourant $1,\ldots,k_n$.

Si un U_n coupait C entre I et son côté opposé I^*, il existerait un continu $L\subset U_n$ unissant les deux autres côtés de C, donc de diamètre $\delta(L)\geqslant 1$, et l'on aurait en vertu de (4) $L\subset S_{n_j}$ pour un $j=1,\ldots,k_n$, d'où $\delta(S_{n_j})\geqslant 1$, contrairement à (3). Il existe donc au moins une composante C_n de $C-U_n$ qui est un continu unissant I et I^*, d'où

$$\delta(C_n)\geqslant 1 \quad \text{pour tout} \quad n=1,2,\ldots \tag{5}$$

Formons de ces continus une suite $\{C_n\}$ de façon que chacun d'eux s'y répète une infinité de fois.

Exemple I. Soient

$$K_0=C, \quad K_{1/n}=C_n\times(1/n) \quad \text{pour} \quad n=1,2,\ldots, \tag{6}$$

$$X=K_0+\sum_{n=1}^{\infty} K_{1/n}, \quad Y=(0,\ 1,\ 1/2,\ 1/3,\ \ldots). \tag{7}$$

Le carré K_0 contenant la limite supérieure des $K_{1/n}$, l'ensemble X est fermé. La fonction

$$f(p)=\begin{cases} 0 & \text{pour} \quad p\in K_0, \\ 1/n & \text{pour} \quad p\in K_{1/n} \end{cases}$$

est continue dans X (comme projection de X sur l'axe des y); on a $f(X)=Y$ d'après (7) et $f^{-1}(1/n)=K_{1/n}$, d'où, en vertu de (5) et (6),

$$\delta[f^{-1}(y)]=\delta(K_{1/n})\geqslant 1 \quad \text{pour} \quad n=1,2,\ldots, \tag{8}$$

et tout $K_{1/n}$, de même que $K_0=f^{-1}(0)$, est un continu.

Supposons qu'il existe dans l'espace une somme finie $G=G_1+\ldots+G_k$ d'ensembles ouverts tels que

$$\delta(G_i)<1/2 \quad \text{et} \quad \bar{G}_i\cdot\bar{G}_{i'}=0 \tag{9}$$

pour $i\neq i'$ parcourant $1,\ldots,k$ et qui satisfasse à (2). Soit

$$\eta=\min_{i\neq i'}[1/8,\ \varrho(\bar{G}_i,\bar{G}_{i'})/3], \tag{10}$$

ϱ désignant la distance. Les ensembles

$$H_i=\sum_p[\varrho(p,G_i)<\eta] \tag{11}$$

sont ouverts et l'on a d'après (9)—(11)

$$\delta(H_i)<1/2+2\eta\leqslant 1/2+2/8=3/4, \tag{12}$$

$$\varrho(\bar{H}_i,\bar{H}_{i'})\geqslant\eta \tag{13}$$

pour $i\neq i'$ parcourant $1,\ldots,k$.

En posant $H=H_1+\ldots+H_k$, on a donc $\varrho(G,C-H)\geqslant\eta$ d'après (11) et on en conclut, en désignant par J le segment $0\leqslant y\leqslant\eta/2$, que

$$G\cdot[(C-H)\times J]=0. \tag{14}$$

Soit V_i l'intersection de H_i avec le plan $y=0$. En posant $V=V_1+\ldots+V_k$, on a donc

$$C-H=C-V. \tag{15}$$

Couvrons $\bar{V}_i$ d'une somme finie de cercles rationnels

$$W_i=\sum_{l=1}^{m_i}Q_{n_l}$$

comme suit:

$$\bar{V}_i\subset W_i, \quad \delta(Q_{n_l})<\eta/3 \quad \text{et} \quad Q_{n_l}\cdot\bar{V}_i\neq 0 \tag{16}$$

pour $l=1,\ldots,m_i$.

Comme $V_i\subset H_i$ par définition, d'où $\varrho(\bar{H}_i,\bar{H}_{i'})\leqslant\varrho(\bar{V}_i,\bar{V}_{i'})$, il vient, en vertu de (10), (12) et (16)

$$\delta(W_i)<3/4+2\eta/3\leqslant 3/4+2/24<1 \quad \text{et} \quad \varrho(\bar{W}_i,\bar{W}_{i'})\geqslant\eta/3$$

pour $i\neq i'$ parcourant $1,\ldots,k$. D'après (3), il existe donc un n_i tel que $W_i=S_{n_i}$ et, d'après (4), un r tel que $V\subset U_r$, d'où en vertu de (15)

$$C-U_r\subset C-H. \tag{17}$$

Vu (10) et la définition de la suite $\{C_n\}$, il existe un indice $s>2/\eta$, donc tel que $1/s\in J$, et pour lequel $C_s=C_r$. Il en résulte, en appliquant successivement (6), la définition du continu C_n, (17) est celle de l'intervalle J, que

$$K_{1/s}=C_s\times(1/s)=C_r\times(1/s)\subset(C-U_r)\times(1/s)\subset(C-H)\times J,$$

d'où, en vertu de (14), $G \cdot K_{1/s} = 0$, c'est-à-dire $G \cdot f^{-1}(1/s) = 0$, contrairement à (2) pour $y = 1/s$.

Exemple II. Pour tout $n = 1, 2, \ldots$, interposons à mi-chemin entre $K_{1/n}$ et $K_{1/(n+1)}$ le carré

$$K_{(2n+1)/2n(n+1)} = C \times ((2n+1)/2n(n+1)). \tag{18}$$

$P_n(p)$ désignant le cône qui a le carré (18) pour base et le point p pour sommet, posons

$$P_{n-} = \sum_{p \in K_{1/n}} P_n(p), \qquad P_{n+} = \sum_{p \in K_{1/(n+1)}} P_n(p), \tag{19 *)}$$

$$X^* = X + \sum_{n=1}^{\infty} (P_{n-} + P_{n+}), \qquad Y^* = \text{ segment } 0 \leqslant y \leqslant 1 \tag{20}$$

et entendons par f^* la projection orthogonale de X^* sur Y^*, donc une fonction continue, telle que $f^*(X^*) = Y^*$ et dont les tranches $f^{*-1}(t)$ pour $t \in Y^*$ sont des coupures de X^* par les plans $y = t$. Pour $y = 0, 1, 1/2, 1/3, \ldots$, elles coïncident avec celles de la fonction f; elles sont donc des continus et leur fixation étant démontrée impossible, il en est donc à plus forte raison de même de celle de la décomposition de X^*.

Pour $1/(n+1) \leqslant t \leqslant 1/n$ les tranches $f^{*-1}(t)$ sont des coupures des sommes (19) de cônes par les plans parallèles à leur base commune (18) ou — ce qui revient au même — les sommes des coupures des $P_n(p)$, donc des continus. En considérant ces continus comme des valeurs de la fonction manifestement continue dont la variable p parcourt un continu d'après (19), la somme $f^{*-1}(t)$ de ces valeurs est donc aussi un continu ([1], p. 123). Comme elle contient la projection orthogonale de $K_{1/n}$ ou de $K_{1/(n+1)}$ (suivant que t dépasse ou non la moyenne de ces indices) sur le plan $y = t$, on a $\delta[f^{*-1}(t)] \geqslant \delta(K_{1/n}) \geqslant 1$ en vertu de (8). Enfin, X^* est un continu en vertu de (20), chacun des solides P_{n+} ayant par définition la base commune avec P_{n-} et tous les sommets communs avec $P_{(n+1)-}$.

INSTITUT MATHÉMATIQUE DE L'ACADÉMIE POLONAISE DES SCIENCES

OUVRAGES CITÉS

[1] K. Kuratowski, *Topologie II*, Monogr. Matem., Warszawa-Wrocław, 1950.

[2] A. Kosiński, *A theorem on monotone mappings*, Bull. Acad. Polon. Sci., Cl. III, **3** (1955), 69-72.

*) Les solides P_{n-} et P_{n+} ont été imaginés par A. Lelek.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

MATHÉMATIQUE

Sur un problème de la théorie de la mesure de H. Steinhaus et S. Ruziewicz

par

S. MARCUS

Présenté par W. SIERPIŃSKI, le 28 Janvier 1956

1. Dans une note publiée dans Fundamenta Mathematicae [1], S. Ruziewicz donne une réponse au problème, posé par H. Steinhaus: *Que sait-on sur un ensemble linéaire, tel qu'aucun point équidistant entre deux points quelconques de cet ensemble ne lui appartient pas?*

S. Ruziewicz obtient la réponse suivante: *La mesure intérieure (au sens de Lebesgue) d'un tel ensemble est nulle.*

Sa démonstration se sert de méthodes liées à l'étude de l'ensemble de distances associé à un ensemble de mesure positive. Du reste, une des conséquences de sa démonstration est le théorème suivant de H. Steinhaus [2]: *L'ensemble de distances d'un ensemble linéaire, de mesure positive, contient un segment ayant l'origine pour l'extrémité gauche.*

Dans la présente note je montrerai que la propriété constatée par S. Ruziewicz s'obtient aussi par une autre voie, d'une manière directe et naturelle, et pour des ensembles plus généraux que ceux du problème de H. Steinhaus. Puis, je démontrerai une autre propriété de ces ensembles, qui constitue l'analogue qualitatif de la propriété constatée par S. Ruziewicz.

2. Je dis qu'un ensemble linéaire E est *totalement asymétrique*, s'il existe — pour chaque point $\xi \in E$ — un intervalle $I=(\xi-\eta, \xi+\eta)$ centré en ξ, de sorte que

$$x \in E \cap I \to 2\xi - x \notin E.$$

THÉORÈME 1. *La mesure intérieure (au sens de Lebesgue) d'un ensemble E totalement asymétrique est égale à zéro.*

Note. Je remercie le Professeur R. Sikorski, pour ses remarques utiles, me permettant d'améliorer sensiblement la rédaction de la présente note.

Démonstration. Puisque un sousensemble d'un ensemble totalement asymétrique est aussi totalement asymétrique, il suffit de démontrer que tout ensemble mesurable, totalement asymétrique, est de mesure nulle.

Soit E un tel ensemble. Soit E' le complémentaire de E. Soit I_x un intervalle quelconque contenant x. On a, conformément au théorème de la densité de Lebesgue, et notant par m la mesure de Lebesgue,

$$\lim_{m(I_x)\to 0} \frac{m(E' \cap I_x)}{m(I_x)} = 0 \quad \text{pour presque tout} \quad x \in E. \tag{1}$$

et

$$\lim_{m(I_x)\to 0} \frac{m(E \cap I_x)}{m(I_x)} = 1 \quad \text{pour presque tout} \quad x \in E. \tag{2}$$

D'autre part, pour des intervalles I_x centrés en x et suffisamment petits, $E' \cap I_x$ contient l'ensemble symétrique de $E \cap I_x$ par rapport à x (puisque E est totalement asymétrique); donc

$$m(E \cap I_x) \leqslant m(E' \cap I_x). \tag{3}$$

De (2) et (3) il résulte

$$\lim_{m(I_x)\to 0} \frac{m(E' I_x)}{m(I_x)} = 1 \quad \text{pour presque tout} \quad x \in E. \tag{4}$$

Il s'ensuit de (1) et (4) que $m(E) = 0$.

3. Voici l'analogue qualitatif du théorème 1:

Théorème 2. *Tout ensemble E, jouissant de la propriété de Baire et contenu dans un ensemble totalement asymétrique, est de première catégorie (au sens de Baire).*

Démonstration. Il suffit de démontrer que tout ensemble totalement asymétrique E, jouissant de la propriété de Baire, est de première catégorie.

Soit E' le complémentaire de E.

Pour des intervalles I_x, centrés en x ($x \in E$) et suffisamment petits, $E' \cap I_x$ contient l'ensemble symétrique de $E \cap I_x$ par rapport à x. Il s'ensuit, en notant par $D(X)$ l'ensemble des points où l'ensemble X est de deuxième catégorie,

$$D(E) \subset D(E'), \quad \text{donc} \quad D(E) \subset D(E) \cap D(E').$$

D'autre part, l'ensemble $D(E) \cap D(E')$ est non dense [3]. E est donc de première catégorie.

COROLLAIRE. *Tout ensemble jouissant de la propriété de Baire et contenu dans une base de Hamel, est de première catégorie.*

INSTITUT DE MATHÉMATIQUE DE L'ACADÉMIE DES SCIENCES DE LA RÉPUBLIQUE POPULAIRE ROUMAINE, BUCAREST.

OUVRAGES CITÉS

[1] S. Ruziewicz, *Contribution à l'étude des ensembles de distances de points*, Fund. Math. **7** (1925), 141.

[2] H. Steinhaus, *Sur les distances des points des ensembles de mesure positive*, Fund. Math. **1** (1920), 93, théorème 8.

[3] K. Kuratowski, *Topologie I*, Varsovie 1948, p. 56.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

MATHÉMATIQUE

Sur un problème de F. Hausdorff concernant les fonctions symétriques continues

par

S. MARCUS

Présenté par W. SIERPIŃSKI, le 28 Janvier 1956

1. Une fonction réelle (finie) $f(x)$, définie dans un intervalle (a,b), est symétriquement continue en un point $x \in (a,b)$, si

$$\lim_{h\to 0}(f(x+h)-f(x-h))=0.$$

Hausdorff a posé en 1935 [1] le problème suivant, consistant en deux questions:

(α) *Etant donné une fonction réelle d'une variable réelle, symétriquement continue en chaque point, est-il possible que l'ensemble de ses points de discontinuité (au sens de Cauchy) soit indénombrable?*

(β) *Étant donné un ensemble linéaire D du type F_σ, existe-t-il une fonction $f(x)$, symétriquement continue en chaque point, tel que D soit l'ensemble de ses points de discontinuité?*

Je n'ai pas rencontré dans la littérature de réponse à ces questions. Cependant, avant 1935, on a obtenu des résultats qui constituent des réponses partielles à la question (α).

Ainsi, comme réponse à un problème posé par H. Steinhaus [2], S. Mazurkiewicz a démontré que, pour toute fonction mesurable, qui admet en chaque point une dérivée symétrique nulle, l'ensemble des points de discontinuité est non-dense [3]. W. Sierpiński a complété le résultat de S. Mazurkiewicz, en montrant, dans les mêmes hypothèses, que l'ensemble des points de discontinuité est dénombrable [4]. Z. Charzyński a démontré que pour une fonction qui admet en chaque point une dérivée symétrique nulle, l'ensemble des points de discontinuité est clairsemé [5]. E. Szpilrajn-Marczewski a démontré que, étant donné un ensemble clairsemé D, il existe une fonction à dérivée symétrique identiquement nulle, continue et nulle pour $x \notin D$, discontinue et positive pour $x \in D$ [6].

En ce qui concerne la question (β), je ne connais que le résultat de H. Fried [7]: *Une fonction symétriquement continue en chaque point est ponctuellement discontinue.*

Vu ce résultat, la question (β) du problème de F. Hausdorff devient:

(β′) *Étant donné un ensemble linéaire E, de première catégorie et du type F_σ, existe-t-il une fonction symétriquement continue en chaque point, telle que l'ensemble de ses points de discontinuité coïncide avec E?*

Le but de la présente note est de trancher par la négative la question (β′); en même temps, je suppose que la méthode appliquée et le résultat acquis contribueront dans une mesure considérable à la solution du problème (α).

Cependant, les ensembles F_σ arbitraires sont liés intimement à la continuité symétrique. Cette assertion est confirmée par le théorème suivant, que je démontre ailleurs [8]: *Etant donné un ensemble linéaire D, du type F_σ, il existe une fonction symétriquement continue aux points et seulement aux points étrangers à D.*

Définition. Une fonction de variable réelle, prenant des valeurs finies ou infinies, est symétriquement continue au sens généralisé au point ξ *) s'il existe un intervalle I centré en ξ, tel que

$$x \in I,\ f(x) = +\infty \to f(2\xi - x) = +\infty,$$

$$x \in I,\ f(x) = -\infty \to f(2\xi - x) = -\infty,$$

$$\lim_{x \to \xi} (f(x) - f(2\xi - x)) = 0, \qquad (-\infty < f(x) < +\infty).$$

Lemme 1. *Si une fonction $f(x)$ réelle (finie) est symétriquement continue au point ξ, alors son oscillation $\omega(x)$ est symétriquement continue au sens généralisé au point ξ.*

Démonstration. Soit $\varepsilon > 0$. Il existe $\eta(\varepsilon) > 0$, tel que

$$|x - \xi| < \eta(\varepsilon) \to |f(x) - f(2\xi - x)| < \varepsilon. \tag{1}$$

Soit $x \in (\xi - \eta, \xi + \eta)$, $x \neq \xi$. Supposons, pour fixer les idées, qu'on ait $\xi < x < \xi + \eta$. Soit un intervalle I contenant x et contenu dans $(\xi, \xi + \eta)$. Soient y et z deux points de I. Soit I' l'intervalle symétrique, par rapport à ξ, de l'intervalle I; soit $x' = 2\xi - x$, $y' = 2\xi - y$, $z' = 2\xi - z$. Tenant compte de (1), il s'ensuit que

$$|f(y) - f(z)| \leqslant 2\varepsilon + |f(y') - f(z')|.$$

*) Nous avons introduit la notion de "fonction symétriquement continue au sens généralisé" à la suite d'une remarque du Prof. R. Sikorski.

Donc, en notant par $\omega(f; I)$ l'oscillation de la fonction $f(x)$ dans I,

$$\omega(f; I) \leqslant \omega(f; I') + 2\varepsilon$$

et

$$\omega(x) \leqslant 2\varepsilon + \omega(x').$$

Tenant compte de ce que dans la dernière relation on peut permuter x et x', il résulte

$$|x-\xi| < \eta(\varepsilon) \rightarrow \begin{array}{ll} |\omega(x)-\omega(x')| \leqslant 2\varepsilon, & \text{si} \quad \omega(x) < +\infty, \\ \omega(x) = \omega(x') = +\infty & \text{si} \quad \omega(x) = +\infty. \end{array}$$

LEMME 2. *L'ensemble des points de discontinuité d'une fonction $f(x)$ ponctuellement discontinue, coïncide avec l'ensemble des points de discontinuité de sa fonction oscillation $\omega(x)$.*

Démonstration. Puisque $f(x)$ est ponctuellement discontinue, l'ensemble des points où $\omega(x)=0$ est dense. Soit ξ un point de discontinuité de $f(x)$. On a $\omega(\xi)>0$, donc $\omega(x)$ est discontinue en ξ.

Soit ξ' un point de discontinuité pour $\omega(x)$. Puisque $\omega(x)$ est semi-continue supérieurement, on a

$$\limsup_{x \to \xi'} \omega(x) \leqslant \omega(\xi').$$

Admettons, pour l'instant, que $\omega(\xi')=0$. Alors, en tenant compte de ce que $\omega(x)$ est non négative, il résulte que $\omega(x)$ est continue en ξ', ce qui est contradictoire.

Donc $\omega(\xi')>0$ et $f(x)$ sont discontinues en ξ'.

COROLLAIRE. *Étant donnée une fonction $f(x)$, ponctuellement discontinue, un point ξ est de discontinuité pour $\omega(x)$, si et seulement si $\omega(\xi)>0$.*

LEMME 3. *L'ensemble triadique de Cantor ne contient aucun point de symétrie* *).

Démonstration. Supposons, par réduction à l'absurde, que l'ensemble $\mathcal{C}$ de Cantor contient un point de symétrie ξ. Il existe donc un intervalle I, centré en ξ, tel que pour $x \in \mathcal{C} \cap I$ on a $2\xi - x \in \mathcal{C}$. Il s'ensuit donc que si J_1 est un intervalle contigu à l'ensemble $\mathcal{C}$ et situé dans I, l'intervalle J_1' symétrique de l'intervalle J_1 par rapport à ξ, est aussi contigu à $\mathcal{C}$. J_1 et J' ont la même longueur, qui est $1/3^n$.

Il résulte du mode de construction de l'ensemble $\mathcal{C}$ que si J_1 et J_1' étaient deux intervalles consécutifs parmi les intervalles de longueur $1/3^n$, contigus à $\mathcal{C}$, alors ξ serait le milieu d'un intervalle contigu à $\mathcal{C}$ (et de longueur $\geqslant 1/3^{n-1}$). Mais cela est contradictoire, puisque $\xi \in \mathcal{C}$. Il existe donc, dans I, notamment entre J_1 et J_1', au moins un intervalle J_2 contigu à $\mathcal{C}$ et de longueur $1/3^n$.

*) Un point $\xi \in \mathcal{G}$ est un point de symétrie de l'ensemble $\mathcal{G}$ s'il existe un intervalle I centré en ξ, tel que $x \in \mathcal{G} \cap I \rightarrow 2\xi - x \in \mathcal{G}$.

Puisque ξ est un point de symétrie pour $\mathcal{C}$, l'intervalle J_2', symétrique avec J_2 par rapport à ξ, est aussi contigu à $\mathcal{E}$.

Utilisant le même raisonnement pour les intervalles J_2 et J_2', on trouve dans I deux nouveaux intervalles J_3 et J_3', contigus à $\mathcal{C}$ et de longueur $1/3^n$, etc.

On forme ainsi une suite infinie d'intervalles disjoints, de la même longueur $1/3^n$, chacun d'eux contenu dans l'intervalle fini I.

La contradiction acquise achève la démonstration.

Lemme 4. *Étant donné un ensemble linéaire $\mathcal{E}$, du type F_σ, s'il existe une fonction $f(x)$, symétriquement continue en chaque point, discontinue aux points et seulement aux points de $\mathcal{E}$, alors il existe une fonction $\omega(x)$ non négative, symétriquement continue au sens généralisé et supérieurement semi-continue en tous les points, positive et discontinue aux points et seulement aux points de $\mathcal{E}$.*

Démonstration. Supposons qu'il existe une fonction $f(x)$ jouissant des propriétés demandées ci-dessus. Vu la proposition 1, l'oscillation $\omega(x)$ de $f(x)$ est symétriquement continue au sens généralisé en chaque point. Vu le théorème cité de H. Fried, la fonction $f(x)$ est ponctuellement discontinue. Vu le lemme 2, $\mathcal{E}$ est l'ensemble des points de discontinuité de $\omega(x)$. Du corollaire du lemme 2, il s'ensuit que $\omega(x)>0$ si $x \in \mathcal{E}$ et $\omega(x)=0$ si $x \notin \mathcal{E}$. Remarquons, enfin, que la fonction $\omega(x)$ est supérieurement semi-continue.

Théorème. *Il n'existe aucune fonction réelle (finie), symétriquement continue en chaque point, discontinue aux points et seulement aux points de l'ensemble triadique $\mathcal{C}$ de Cantor.*

Démonstration. Supposons, par réduction à l'absurde, qu'il existe une telle fonction $f(x)$. Vu le lemme 4, il existe une fonction $\omega(x)$ non négative, symétriquement continue au sens généralisé et supérieurement semi-continue en chaque point, positive et discontinue aux points et seulement aux points de $\mathcal{C}$.

Il peut arriver deux cas:

a) $\omega(x)=+\infty$ pour tout $x \in \mathcal{C}$;

b) Il existe $x \in \mathcal{C}$, tel que $\omega(x)<+\infty$.

Il existe donc un intervalle I ($I \cap \mathcal{C} \neq 0$), où $\omega(x)$ est finie. $\omega(x)$ est de première classe dans I et, conformément au théorème de Baire, il existe $\xi \in I \cap \mathcal{C}$, tel que $\omega(x)$ est continue en ξ, par rapport à $\mathcal{C}$.

En tout cas, il existe un point $\xi \in \mathcal{C}$, tel que

$$\lim_{x \to \xi,\, x \in \mathcal{C}} \omega(x) = \omega(\xi). \tag{2}$$

Nous allons montrer que le point ξ est un point de symétrie pour $\mathcal{C}$, ce qui contredit le lemme 3 et démontre le théorème.

En effet, dans le cas contraire il existerait une suite $\{x_n\}$, $x_n \in \mathcal{C}$, $x_n \to \xi$, de sorte que pour la suite symétrique $\{2\xi - x_n\}$ on aurait $2\xi - x_n \notin \mathcal{C}$. Mais, tenant compte de (2), on a

$$\lim_{n\to\infty} \omega(x_n) = \omega(\xi).$$

En même temps il résulte de $2\xi - x_n \notin \mathcal{C}$ que $\omega(2\xi - x_n) = 0$ pour $n = 1, 2, \ldots$; donc

$$\lim_{n\to\infty} \omega(2\xi - x_n) = 0 \neq \omega(\xi).$$

Les deux relations acquises sont en contradiction avec la continuité symétrique au sens généralisé de $\omega(x)$ au point ξ.

Donc ξ est un point de symétrie pour $\mathcal{C}$.

Remarque. Dans un ordre d'idées analogue on a les théorèmes suivants établis par W. Sierpiński [9]:

1. Il existe une fonction $f(x)$ d'une variable réelle non mesurable et partout presque symétrique.

2. Il existe une fonction non mesurable $f(x)$ qui est partout approximativement symétrique au sens large.

3. Il existe une fonction non mesurable $f(x)$ qui a partout la derivée symétrique approximative au sens large égale à 0.

4. Si $2^{\aleph_0} = \aleph_1$, il existe une fonction non mesurable $f(x)$ qui a partout la derivée symétrique approximative au sens restreint égale à 0.

INSTITUT DE MATHÉMATIQUE DE L'ACADÉMIE DES SCIENCES DE LA RÉPUBLIQUE POPULAIRE ROUMAINE, BUCAREST

OUVRAGES CITÉS

[1] F. Hausdorff, Problème No. 62, Fund. Math. **25** (1935), 578.

[2] H. Steinhaus, Problème No. 23, Fund. Math. **4** (1923), 368.

[3] S. Mazurkiewicz, *Sur la dérivée première généralisée*, Fund. Math. **11** (1928), 145.

[4] W. Sierpiński, *Sur une hypothèse de S. Mazurkiewicz*, Fund. Math. **11** (1928), 148.

[5] Z. Charzyński, *Sur les fonctions dont la dérivée symétrique est partout finie*, Fund. Math. **21** (1933), 214.

[6] E. Szpilrajn-Marczewski, *Remarque sur la dérivée symétrique*, Fund. Math. **21** (1933), 226.

[7] H. Fried, *Sur les fonctions symétriquement continues*, Fund. Math. **29** (1937), 134.

[8] S. Marcus, *Les ensembles F_σ et la continuité symétrique*, Bull. St. Acad. R. P. R., série mathématique (1955).

[9] W. Sierpiński, *Sur une fonction non mesurable, partout presque symétrique*, Acta Scient. Math., Szeged, **8** (1936), 1.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

MATHEMATICS

A Note on Some Properties of the Functions $\varphi(n)$, $\sigma(n)$ and $\theta(n)$

by

A. SCHINZEL and Y. WANG

Presented by W. SIERPIŃSKI on February 2, 1956

A. Schinzel has proved in [1] that for every sequence of h positive numbers $a_1, \ldots, a_h$ there exist infinite sequences of natural numbers n_k and n'_k, such that

$$\lim_{k=\infty} \frac{\varphi(n_k+i)}{\varphi(n_k+i-1)} = \lim_{k=\infty} \frac{\sigma(n'_k+i)}{\sigma(n'_k+i-1)} = a_i \quad (1 \leqslant i \leqslant h).$$

We shall give here certain stronger theorems, which are deduced from the following theorem.

THEOREM 1. *Let*

$$A_0 = h! q_1 \cdot \ldots q_s \cdot q_{0,1} \cdot \ldots \cdot q_{0,t_0}$$
$$A_1 = q_{i,1}, \ldots \cdot q_{i,t_i} \ (1 \leqslant i \leqslant h)$$

be given natural numbers where $q_1, q_2, \ldots, q_s$ are all prime numbers in the interval $0 < x \leqslant 10(h+1)$ and where $q_{i,j}$ $(0 \leqslant i \leqslant h,\ 1 \leqslant j \leqslant t_i)$ are prime numbers greater than $10(h+1)$ such that $A_0, A_1, \ldots, A_h > 1$ are relatively prime in pairs.

If $Z > A_0 A_1^2 \cdot \ldots \cdot A_h^2$, and $N_Z(X)$ denotes the number of positive solutions $(x_0, \ldots, x_h)$ of the system of equations

$$A_0 x_0 + i = i A_i x_i \ (1 \leqslant i \leqslant h)$$

satisfying the conditions

$$1 \leqslant x_0 \leqslant X, \quad \text{if} \quad p \mid x_i \quad \text{then} \quad p > Z \quad (0 \leqslant i \leqslant h),$$

where p is a prime, then there exist positive constants c_1, dependent only on h, and c_2, X_1, dependent only on A_i, such that

$$N_X c_1(X) > \frac{c_2 X}{\log^{h+1} X} \quad (X > X_1).$$

If $Z > A_0 A_1^2 \cdot \ldots \cdot A_h^2$, λ is a given positive number in the interval $0 \leqslant \lambda < A_0 A_1^2 \cdot \ldots \cdot A_h^2$ and $p_1 < p_2 \ldots < p_r$ are all prime numbers that do not divide $A_0 A_1 \cdot \ldots \cdot A_h$ and do not exceed Z, and if $a_{j,i}$ $(1 \leqslant j \leqslant r,\ 0 \leqslant i \leqslant h)$ are given positive numbers satisfying the conditions $0 \leqslant a_{j,i} < p_j$ $(1 \leqslant j \leqslant r,\ 0 \leqslant i \leqslant h)$ and $i_1 \neq i_2$ implies $a_{j,i_1} \neq a_{j,i_2}$ $(1 \leqslant j \leqslant r)$, then we can define $M_Z(x)$ as the number of numbers x satisfying the conditions

$$1 \leqslant x \leqslant X, \quad x \equiv \lambda \pmod{A_0 A_1^2 \cdot \ldots \cdot A_h^2}, \quad x \not\equiv a_{j,i} \pmod{p_j} \quad (1 \leqslant j \leqslant r,\ 0 \leqslant i \leqslant h).$$

Theorem 1 obviously results from the following two lemmas:

LEMMA 1. *There exist λ and $a_{j,i}$ such that $N_Z(X) \geqslant M_Z(X)$.*

LEMMA 2. *There exist positive constants c_1, dependent only on h, and c_2, X_1, dependent only on A_i, such that*

$$M_X c_1(X) \geqslant \frac{c_2 X}{\log^{h+1} X} \quad (X \geqslant X_1)$$

for any given λ and $a_{j,i}$.

Lemma 1 can be deduced from Lemma 2 [1]; Lemma 2 is obtained by Brun's method [4].

From Theorem 1 follows

THEOREM 2. *For any given sequence of h non-negative numbers $a_1, a_2, \ldots, a_h$ and $\varepsilon > 0$, there exists a natural number n such that*

$$(*) \qquad \left| \frac{\varphi(n+i)}{\varphi(n+i-1)} - a_i \right| < \varepsilon \quad (1 \leqslant i \leqslant h).$$

There exist positive constants $c = c(a, \varepsilon)$ and $X_0 = X_0(a, \varepsilon)$ such that the number of numbers n satisfying (*) in the interval $1 \leqslant n \leqslant X$ is greater than $cX/\log^{h+1} X$ for $X > X_0$.

Professor Hua Loo-Keng was the first to point out the possibility of proving this by means of Brun's method.

THEOREM 3 is obtained from Theorem 2 by replacing the letters φ by σ, c by c', X_0 by X_0'.

Further, we have

THEOREM 4. *For any given natural number h there exists a constant b such that for any given sequence of h integers $a_1, a_2, \ldots, a_h > 1$ there exists a natural number n such that*

$$(**) \qquad \theta(i) a_i < \theta(n+i) < b\,\theta(i) a_i \quad (1 \leqslant i \leqslant h).$$

There exist positive constants $c'' = c''(a)$ and $X_0'' = X_0''(a)$ such that the number of numbers n satisfying (**) in the interval $1 \leqslant n \leqslant X$ is greater than $c'' X/\log^{h+1} X$ for $X > X_0''$.

THEOREM 5. *For any given sequence of h numbers $a_1, \ldots, a_h$, where $a_i = 0$ or $+\infty$ $(1 \leqslant i \leqslant h)$, there exists an infinite sequence of natural num-*

bers $n_1, n_2, \ldots$ *such that*

$$\lim_{k=\infty} \frac{\theta(n_k+i)}{\theta(n_k+i-1)} = a_i \qquad (1 \leqslant i \leqslant h).$$

The last theorem gives a positive solution to the problem proposed in paper [2] of A. Schinzel and comprises the theorem from paper [3] of A. Schinzel.

The question whether a theorem analogous to Theorems 2 and 3 is true for the function $\theta(n)$ remains open.

Detailed proofs of the above theorems will appear in Annales Polonici Mathematici.

In the course of publication of the present paper, Theorems 3 and 5 were proved also by Shao Pin Tsung independently (to appear in Shou Hsueh Chin Chan (Progress of Mathematics) and Pei-Ta Hsueh Pao (Transactions of Peking University)).

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES
INSTITUTE OF MATHEMATICS, ACADEMIA SINICA

REFERENCES

[1] A. Schinzel, *On functions $\varphi(n)$ and $\sigma(n)$*, Bull. Acad. Polon. Sci., Cl. III, **3** (1955), 415.

[2] — *Quelques théorèmes sur les fonctions $\varphi(n)$ and $\sigma(n)$*, Bull. Acad. Polon. Sci., Cl. III, **2** (1954), 467.

[3] — *Sur une propriété du nombre de diviseurs*, Publ. Math. Debrecen **3** (1954), 261.

[4] E. Rademacher, *Beiträge zur Viggo Brunschen Methode in der Zahlentheorie*, Abhandlungen aus dem Matematischen Seminar, Hamburg, 1923.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

MATHEMATICS

On a Characterisation of Connected Domains in Locally Connected Spaces

by

K. KURATOWSKI

Presented on February 10, 1956

By definition ([4], p. 43, and [1], p. 176), an open set D is called an (open) *domain* (or a "regular" open set), when it is the set of interior points of a closed set; otherwise stated — when it is the interior of its closure:

(i) $$\operatorname{Int}(\overline{D})=D,$$

or — what is equivalent — when *)

(ii) $$\operatorname{Fr}(D)\subset\operatorname{Fr}(\overline{D}).$$

The purpose of this paper is to characterise connected domains lying in locally connected spaces in terms of their boundaries. An invariance theorem on connected domains lying in Euclidean spaces will follow.

1. Properties of l. c. (= locally connected) spaces

We assume in this section that the given space $\mathfrak{X}$ is l. c. Let us recall some known properties of sets lying in an l. c. space $\mathfrak{X}$:

I. $$\operatorname{Fr}\Big(\sum_t X_t\Big)\subset\overline{\sum_t \operatorname{Fr}(X_t)},$$

where t is arbitrary parameter ([5], p. 168).

II. *If X is a component* **) *of Y, then* $\operatorname{Fr}(X)\subset\operatorname{Fr}(Y)$ ([5], p. 169, 3).

*) We denote by $\operatorname{Int}(X)$ the interior of the set X (lying in the space $\mathfrak{X}$) and by $\operatorname{Fr}(X)$ — its boundary:

$$\operatorname{Int}(X)=\mathfrak{X}-\overline{\mathfrak{X}-X},\qquad \operatorname{Fr}(X)=\overline{X}\cdot\overline{\mathfrak{X}-X}.$$

**) i. e. a connected subset of Y which is not contained in any other connected subset of Y.

Moreover:

III. *If $\{X_t\}$ is the set of all components of Y, then*

$$\mathrm{Fr}(Y)=\overline{\sum_t \mathrm{Fr}(X_t)}.$$

Proof (see also [3], p. 155 (7)). As $Y=\sum_t X_t$, we have by I

$$\mathrm{Fr}(Y)=\mathrm{Fr}\Big(\sum_t X_t\Big)\subset\overline{\sum_t \mathrm{Fr}(X_t)}.$$

Conversely, we have by II

$$\mathrm{Fr}(X_t)\subset\mathrm{Fr}(Y), \quad \text{hence} \quad \overline{\sum_t \mathrm{Fr}(X_t)}\subset\overline{\mathrm{Fr}(Y)}=\mathrm{Fr}(Y).$$

IV. *If R is a region (= connected open set) and C a component of $\mathfrak{X}-\bar{R}$, then $\mathrm{Fr}(C)$ is an irreducible cutting of X between two points* *) ([5], p. 175, 2).

V. *Each irreducible cutting of $\mathfrak{X}$ between two points is the common boundary of two disjoint regions* ([5], p. 175, 1).

2. Domains in l. c. spaces

THEOREM 1. *Every component R of a domain D is a domain.*

Proof. As D is open, so is R ([5], p. 163, 4). Hence

$$\mathrm{Int}(\bar{R})\cdot\mathrm{Fr}(R)=\mathrm{Int}(\bar{R})\cdot\bar{R}-R=\mathrm{Int}(\bar{R})-R. \tag{1}$$

By II, $\mathrm{Fr}(R)\subset\mathrm{Fr}(D)$. Therefore

$$\mathrm{Int}(\bar{R})\cdot\mathrm{Fr}(R)\subset\mathrm{Int}(\bar{D})\cdot\mathrm{Fr}(D)=\mathrm{Int}(\bar{D})-D. \tag{2}$$

But $\mathrm{Int}(\bar{D})-D=0$ by (i), and it follows from (1) and (2) that $\mathrm{Int}(\bar{R})-R=0$; whence, according to (i), R is a domain.

THEOREM 2. *If D is a domain lying in an l. c. space $\mathfrak{X}$, then*

$$\mathrm{Fr}(D)=\overline{\sum_t I_t}, \tag{3}$$

where I_t is an irreducible cutting of $\mathfrak{X}$ between two points.

More precisely: *If the formulae*

$$D=\sum_u R_u \tag{4}$$

*) An irreducible cutting of $\mathfrak{X}$ between two points a and b is a closed set which separates $\mathfrak{X}$ between a and b and whose no proper closed subset separates $\mathfrak{X}$ between those points.

and

$$\mathfrak{X}-\bar{R}_u=\sum_v C_{u,v} \tag{5}$$

represent the decomposition of the sets D *and* $\mathfrak{X}-\bar{R}_u$ *into their components, then*

$$\mathrm{Fr}(D)=\overline{\sum_{u,v}\mathrm{Fr}(C_{u,v})}, \tag{6}$$

(*and* $\mathrm{Fr}(C_{u,v})$ *is an irreducible cutting between two points by* IV).

Proof. According to (4), I, Th. 1 and (ii), we have

$$\mathrm{Fr}(D)=\mathrm{Fr}\Big(\sum_u R_u\Big)\subset\overline{\sum_u \mathrm{Fr}(R_u)}\subset\overline{\sum_u \mathrm{Fr}(\bar{R}_u)}. \tag{7}$$

But $\mathrm{Fr}(\bar{R}_u)=\mathrm{Fr}(\mathfrak{X}-\bar{R}_u)$; hence, by III and (5)

$$\mathrm{Fr}(\bar{R}_u)=\overline{\sum_v \mathrm{Fr}(C_{u,v})}. \tag{8}$$

It follows from (7) and (8) that

$$\mathrm{Fr}(D)\subset\overline{\sum_u\overline{\sum_v \mathrm{Fr}(C_{u,v})}}\subset\overline{\sum_u\sum_v \mathrm{Fr}(C_{u,v})}.$$

Conversely, we conclude from II that (5) and (4) imply

$$\mathrm{Fr}(C_{u,v})\subset\mathrm{Fr}(\mathfrak{X}-\bar{R}_u)=\mathrm{Fr}(\bar{R}_u)\subset\mathrm{Fr}(R_u)\subset\mathrm{Fr}(D).$$

Therefore

$$\overline{\sum_{u,v}\mathrm{Fr}(C_{u,v})}\subset\mathrm{Fr}(D),$$

completing the proof of (6).

THEOREM 3. *Let* D *be a region in an l. c. space* $\mathfrak{X}$. *Then the condition* (3) *is necessary and sufficient in order that* D *be a domain.*

Proof. The condition is necessary by Theorem 2. In order to show its sufficiency, let us assume that the region D satisfies to (3). We have to show that (ii) is satisfied, i. e. that $\mathrm{Fr}(D)\subset\overline{\mathfrak{X}-\bar{D}}$, or — according to (3) — that

$$I_t\subset\overline{\mathfrak{X}-\bar{D}}. \tag{9}$$

Now, by V there exist two disjoint open sets U and V such that

$$\mathrm{Fr}(U)=I_t=\mathrm{Fr}(V). \tag{10}$$

By (3), we have $I_t\subset\mathrm{Fr}(D)$; hence $DI_t=0$. Since D is connected and U and V are disjoint open sets, one of the formulae: $UD=0$ or $VD=0$, holds true. For, otherwise, we would have $D\cdot\mathrm{Fr}(U)\neq 0$, i. e. $DI_t\neq 0$.

Thus we may assume that $VD=0$. It follows that $V\cdot\bar{D}=0$ (since V is open), i. e. $V\subset\mathfrak{X}-\bar{D}$; hence by (10), $I_t\subset\bar{V}\subset\overline{\mathfrak{X}-\bar{D}}$.

3. Applications to Euclidean space

Let us denote by $\mathcal{S}_n$ the Euclidean n-dimensional space + the point in infinity. Theorem 3 implies the following invariance theorem.

THEOREM 4. *Let D and E be two regions in $\mathcal{S}_n$ whose boundaries are homeomorphic. If D is a domain, so is E.*

Proof. Let h be a homeomorphic mapping of $\mathrm{Fr}(D)$ onto $\mathrm{Fr}(E)$. D being a domain, formula (3) holds true. Therefore

$$\mathrm{Fr}(E)=h[\mathrm{Fr}(D)]=h\Big(\overline{\sum_t I_t}\Big)=\overline{h\Big[\sum_t (I_t)\Big]}=\overline{\sum_t h(I_t)}.$$

Since the property of being an irreducible cutting of $\mathcal{S}_n$ between two points is an invariant of homeomorphic mappings ([5], p. 349, IV, and [2], p. 104), $h(I_t)$ is an irreducible cutting of $\mathcal{S}_n$ between two points, and the formula above shown,

$$\mathrm{Fr}(E)=\overline{\sum_t h(I_t)},$$

yields the conclusion – according to Theorem 3 – that E is a domain.

THEOREM 5. *Let C be a locally connected continuum lying in the plane $\mathcal{S}_2$. If $\mathcal{S}_2-C$ is a domain, then*

$$\mathrm{Fr}(C)=\overline{\sum_n K_n}, \tag{11}$$

where K_n is a simple closed curve.

Proof. Put $D=\mathcal{S}_2-C$ and apply Theorem 2. C being l. c., so is $\bar{R}_u$ ([5], p. 360, 4 (iii)), and consequently $\mathrm{Fr}(C_{u,v})$ is l. c. ([5], p. 360, 4 (i)). Thus, $\mathrm{Fr}(C_{u,v})$ is a locally connected irreducible cutting of $\mathcal{S}_2$ between two points, hence ([5], p. 356, 8) a simple closed curve. Formula (11) follows at once from (6) applying the identity $\mathrm{Fr}(C)=\mathrm{Fr}(\mathcal{S}_2-C)$.

INSTITUTE OF MATHEMATICS, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] G. Birkhoff, *Lattice Theory*, Coll. Publ. **25**, New York 1948.

[2] S. Eilenberg, *Transformations continues en circonférence et la topologie du plan*, Fund. Math. **26** (1936), 61.

[3] F. Hausdorff, *Mengenlehre*, Berlin-Leipzig 1927.

[4] K. Kuratowski, *Topologie I*, Monogr. Mat. **20**, Warszawa-Wrocław 1948.

[5] – *Topologie II*, Monogr. Mat. **21**, Warszawa-Wrocław 1950.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

On an Operational Method of Solving the Klein-Gordon Equation

by

L. INFELD and J. PLEBAŃSKI

Presented by L. INFELD on February 15, 1956

1. Introduction

The Klein-Gordon equation (for the Green functions) in the usual notation is:

$$(\Box-\mu^2)G=\left(\Delta-\frac{\partial^2}{\partial x_0^2}-\mu^2\right)G=-\delta_3(\vec{x})\,\delta(x_0). \tag{1.1}$$

This equation can be written in either of two forms:

$$\left(\frac{d^2}{dx_0^2}+n^2\right)G=\delta_{(3)}(\vec{x})\,\delta(x_0); \qquad n^2=\mu^2-\Delta, \tag{1.2a}$$

$$(\Delta-m^2)G=-\delta_{(3)}(\vec{x})\,\delta(x_0); \qquad m^2=\mu^2+\frac{\partial^2}{\partial x_0^2}. \tag{1.2b}$$

Thus the form (1.2) in which the equations were written suggests the method of their interpretation. The equation (1.2a) reminds us of that of an oscillator with a natural frequency n; the equation (1.2b) reminds us of the Yukawa equation with mass m. These are the analogies which we shall use looking in turn for the solutions of (1.2a) and (1.2b).

2. The oscillator equation method

In the case of a constant n^2 we know well what n means. What does it mean in the case of an operator? We must connect a definite meaning with

$$n=\sqrt{\mu^2-\Delta} \tag{2.1}$$

We shall define (2.1) through operations on the Fourier transform of a function

$$F(n)f(\vec{x})=F(n)\int d_{(3)}k e^{i\vec{k}\vec{x}}g(\vec{k})=\int d_{(3)}k e^{i\vec{k}\vec{x}}F(\sqrt{\mu^2+k^2})g(\vec{k}). \tag{2.2}$$

It is obvious that this definition preserves the principal property of n, that is

$$(\sqrt{\mu^2-\Delta})^2=n^2=\mu^2-\Delta. \tag{2.3}$$

Now, if n were a constant, the general solution of (1.2a) would have been:

$$G=\cos(n\,x_0)\,a+\sin(n\,x_0)\,b+\frac{1}{2}\,\frac{x_0}{|x_0|}\,\frac{\sin(nx_0)}{n}\,\delta_3(\vec{x}). \tag{2.4}$$

The last (third) expression in (2.4) is a particular solution of the non-homogenous equation (1.2a) since the derivative of $\frac{1}{2}\,x_0/|x_0|$ is equal to $\delta(x_0)$. The first two expressions are added solutions of the homogenous equation. Thus (2.4) represents the most general solution of (1.2a) with constant n, since it has two arbitrary constants.

In the case in which n is the operator (2.1), the same expression (2.4) is the most general solution with one additional remark: a and b are arbitrary functions of $\vec{x}$ and must be written after $\dfrac{\cos(nx_0)}{\sin(nx_0)}$ and not before.

From (2.4) we can easily find the solutions of (1.2a) belonging to the retarded (Δ^{ret}) or advanced (Δ^{adv}) potential. Indeed, Δ^{ret} has to fulfil the following conditions: it has to be a solution of (1.2a) and has to be equal to zero for $x_0<0$. Thus, putting in (2.4)

$$a=0,\quad b=\frac{1}{2n}\,\delta_3(\vec{x}),$$

we have

$$\Delta^{\text{ret}}=\frac{\sin nx_0}{2n}\left(1+\frac{x_0}{|x_0|}\right)\delta_3(\vec{x}). \tag{2.5a}$$

Similarly, we find Δ^{adv} as a solution of (1.2a), vanishing for $x_0>0$

$$\Delta^{\text{adv}}=\frac{\sin nx_0}{2n}\cdot\left(-1+\frac{x_0}{|x_0|}\right)\delta_3(\vec{x}). \tag{2.5b}$$

3. The Yukawa equation method

As before, we start by connecting a definite meaning with:

$$m=\sqrt{\mu^2+\frac{\partial^2}{\partial x_0^2}}. \tag{3.1}$$

Our general definition through Fourier-Transforms is

$$(3.2)\qquad F(m)f(x_0)=F(m)\int_{-\sim}^{+\sim}dve^{-ivx_0}g(v)=$$

$$=\int_{-\sim}^{-|\mu|}+\int_{|\mu|}^{\infty}dve^{-ivx_0}F(-i\xi\sqrt{v^2-\mu^2})g(v)+$$

$$+\int_{-|\mu|}^{|\mu|}dve^{-ivx_0}F(\eta\sqrt{\mu^2-v^2})g(v),$$

where $\sqrt{v^2-\mu^2}$ and $\sqrt{\mu^2-v^2}$ must be understood as positive, and $\xi(v)$, $\eta(v)$ as taking only the values $+1$ or -1. The values for $\xi(v)$ and $\eta(v)$ can be determined by two correspondence principles. Let us start with the value $\xi(v)$. We can find it by taking the limit $\mu\to 0$, that is by the condition

$$\lim_{\mu\to 0}\sqrt{\mu^2+\frac{\partial^2}{\partial^2 x_0^2}}=\frac{\partial}{\partial x_0}.$$

In this case we have confronting the definition of $F\left(\frac{\partial}{\partial x_0}\right)$ with (3.2):

$$(3.3)\qquad F\left(\frac{\partial}{\partial x_0}\right)f(x_0)=F\left(\frac{\partial}{\partial x_0}\right)\int_{-\sim}^{+\sim}dve^{-ivx_0}g(v)=$$

$$=\int_{-\sim}^{+\sim}dve^{-ivx_0}F(-iv)g(v)=\int_{-\sim}^{+\sim}dve^{-ivx_0}F\left(-i\frac{v}{|v|}\sqrt{v^2}\right)g(v),$$

from which follows

$$(3.4)\qquad \xi=\frac{v}{|v|}.$$

Similarly, we shall find the value for η. Putting $x_0=ct$, we have

$$(3.5)\qquad F(m)f(t)=F\left(\sqrt{\mu^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}\right)\int_{-\sim}^{+\sim}d\omega e^{-i\omega t}g(\omega)=$$

$$=\int_{-\sim}^{-|\mu|c}+\int_{|\mu|c}^{\sim}d\omega e^{-i\omega t}F\left(-i\xi\sqrt{\frac{\omega^2}{c^2}-\mu^2}\right)g(\omega)+$$

$$+\int_{-|\mu|c}^{+|\mu|c}d\omega e^{-i\omega t}F\left(\eta\sqrt{\mu^2-\frac{\omega^2}{c^2}}\right)g(\omega).$$

Let us now assume that $c\to\infty$ in (3.5); we obtain

$$F(\sqrt{\mu^2})f(t)=\int_{-\sim}^{+\sim} d\omega e^{-i\omega t}F(\eta|\mu|)g(\omega). \tag{3.6}$$

Therefore to $\sqrt{\mu^2}=+|\mu|$ corresponds $\eta=1$ and to $\sqrt{\mu^2}=-|\mu|$ corresponds $\eta=-1$. Thus we must distinguish between $m_{(+)}=\sqrt{\mu^2+\frac{\partial^2}{\partial x_0^2}}$ with the property

$$\lim_{c\to\infty} m_{(+)}=\lim_{c\to\infty}\sqrt{\mu^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}=+|\mu|$$

and $m_{(-)}$, for which

$$\lim_{c\to\infty} m_{(-)}=\lim_{c\to\infty}\sqrt{\mu^2+\frac{1}{c^2}\frac{\partial^2}{\partial t^2}}=-|\mu|.$$

We have also, according to (3.2) and (3.4)

$$F[m_{(+)}]f(x_0)=F[m_{(+)}]\int_{-\sim}^{+\sim} dve^{-ivx_0}g(v)= \tag{3.7a}$$

$$=\int_{-\sim}^{-|\mu|}+\int_{|\mu|}^{\sim} dve^{-ivx_0}F\left(-i\frac{v}{|v|}\sqrt{v^2-\mu^2}\right)g(v)+$$

$$+\int_{-|\mu|}^{|\mu|} dve^{-ivx_0}F(\sqrt{\mu^2-v^2})g(v).$$

$$F[m_{(-)}]f(x_0)=F[m_{(-)}]\int_{-\sim}^{+\sim} dve^{-ivx_0}g(v)= \tag{3.7b}$$

$$=\int_{-\sim}^{-|\mu|}+\int_{|\mu|}^{\sim} dve^{-ivx_0}F\left(-i\frac{v}{|v|}\sqrt{v^2-\mu^2}\right)g(v)+$$

$$+\int_{-|\mu|}^{|\mu|} dve^{-ivx_0}F(-\sqrt{\mu^2-v^2})g(v).$$

We have

$$(m_{(+)})^2=(m_{(-)})^2=m^2=\mu^2+\frac{\partial^2}{\partial x_0^2}; \tag{3.8}$$

by m we can also understand either $m_{(+)}$ or $m_{(-)}$.

We can now write down the general solution of (1.2b) fulfilling the condition of spherical symmetry:

$$G=\frac{sh(rm_{(+)})}{r}a_{(+)}+\frac{sh(rm_{(-)})}{r}a_{(-)}+\frac{1}{4\pi}\frac{ch(rm)}{r}\delta(x_0). \tag{3.9}$$

The last (third) expression, depending only on m^2, is a particular solution of the non-homogenous equation (1.1), and the first two expressions are solutions of the homogenous equations. Here a are arbitrary functions of x_0.

Let us now restrict the solutions (3.9) to such that they vanish at infinity (that is for $r \to \infty$).

We see that (3.9) satisfying these conditions are:

$$G = \frac{1}{4\pi} \frac{e^{-rm_{(+)}}}{r} b_{(+)} + \frac{1}{4\pi} \frac{e^{rm_{(-)}}}{r} b_{(-)}, \qquad b_{(+)} + b_{(-)} = \delta(x_0). \tag{3.10}$$

We can finally find, as we did before, Δ^{ret} and Δ^{adv}. This can be done in the following way: we are looking for such a solution among (3.10) which has the following property: for $\mu \to 0$ it goes over into D^{ret} and D^{adv}, that is over into

$$\frac{\delta(x_0 - r)}{4\pi r} \quad \text{and} \quad \frac{\delta(x_0 + r)}{4\pi r}.$$

It can be shown, indeed, that these conditions are satisfied by:

$$\Delta^{\text{ret}} = \frac{1}{4\pi r} e^{-rm_{(+)}} \delta(x_0), \tag{3.11a}$$

$$\Delta^{\text{adv}} = \frac{1}{4\pi r} e^{+rm_{(-)}} \delta(x_0), \tag{3.11b}$$

respectively. In fact, putting $m_{(+)} = m_{(-)} = \dfrac{\partial}{\partial x_0}$, we have from (3.11):

$$D^{\text{ret}} = \frac{1}{4\pi r} e^{-r\frac{\partial}{\partial x_0}} \delta(x_0) = \frac{1}{4\pi r} \sum_{l=0}^{\infty} \frac{(-)^l r^l}{l!} \delta^{(l)}(x_0) = \frac{\delta(x_0 - r)}{4\pi r}, \tag{3.12a}$$

$$D^{\text{adv}} = \frac{1}{4\pi r} e^{+r\frac{\partial}{\partial x_0}} \delta(x_0) = \frac{1}{4\pi r} \sum_{l=0}^{\infty} \frac{r^l}{l!} \delta^{(l)}(x_0) = \frac{\delta(x_0 + r)}{4\pi r}. \tag{3.12b}$$

Thus (3.11) is identical in contents, though not in its mathematical form with (2.5).

A more detailed report dealing with the problem of the Klein-Gordon equation will appear in Acta Physica Polonica.

INSTITUTE OF PHYSICS, POLISH ACADEMY OF SCIENCES

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

CHEMISTRY

On the Synthesis and Dégradation of Certain Derivatives of Tetrahydro-1,3-Oxazine. II

by

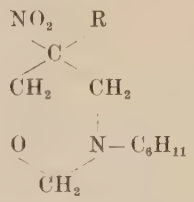
D. GÜRNE and T. URBAŃSKI

Communicated by T. URBAŃSKI at the meeting of November 28, 1955

In the previous paper [1] a number of homologous 5-nitro-5-alkyl-3-benzyltetrahydro-1,3-oxazines were prepared, their hydrolysis and degradation being studied.

In the present work, by using nitroparaffins from nitroethane to 1-nitro-*n*-butane and formaldehyde, and cyclohexylamine, a number of new compounds of the general formula (I) were obtained:

all of them were 5-nitro-5-alkyl-3-cyclohexyl-tetrahydro-1,3-oxazines.



(I) (a) $R = CH_3$
(b) $R = C_2H_5$
(c) $R = C_3H_7$

Compounds (Ia), (Ib) and (Ic) were respectively prepared from nitroethane, 1-nitropropane and 1-nitro-*n*-butane.

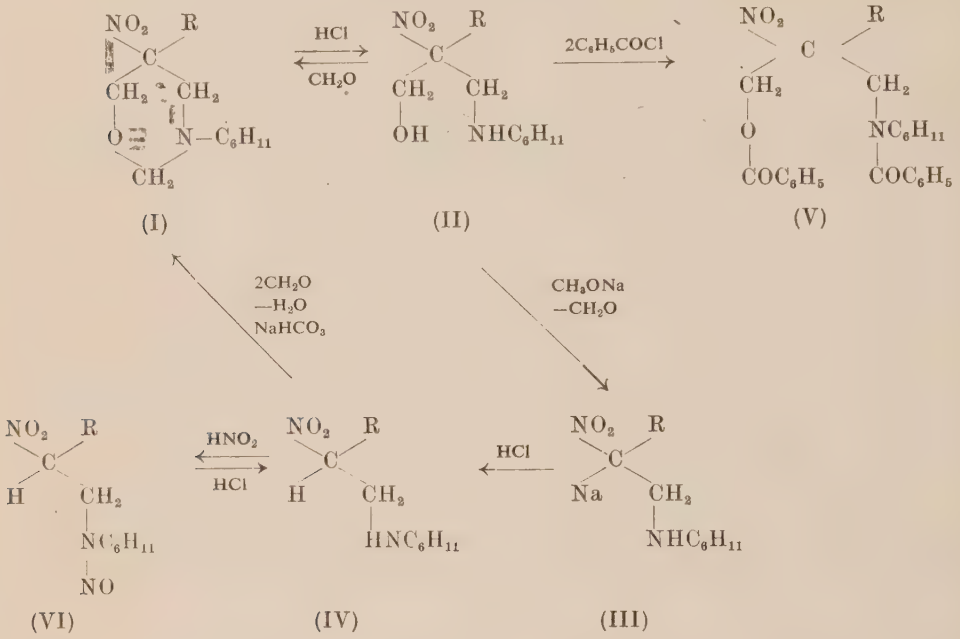
Compound (Ia) had previously been obtained and described by Senkus [2].

All compounds (Ia, b, c) were subjected to degradation in much the same way as previously described [1].

(1) By heating with hydrochloric acid diluted with alcohol to 1.5% HCl, hydrolysis occurred with the splitting off of a molecule of formaldehyde. The open chain aminoalcohols (II) were obtained in this way. Warming of compounds (II) with formaldehyde yielded the starting ring compounds (I).

(2) By treating the aminoalcohols (II) with sodium methoxyde in methanolic solution, a mol. of formaldehyde was cloven off and sodium

salts of aminonitrocompounds (III) resulted. These, upon action of alcoholic HCl, followed by evaporation of the solvent under reduced pressure, were transformed into hydrochlorides of (IV). The hydrochlorides of (IV) yielded, when warmed with formaldehyde in the presence of sodium-hydrogen carbonate, the initial substances (I).

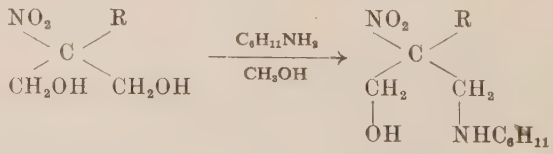


(I) (II) (V)

(VI) (IV) (III)

The structure of compounds (IIa, b, c) and (IVa, b, c) was established: (1) by analysis; (2) by the formation of dibenzoylderivatives (V) of (II); (3) by the formation of crystalline N-nitrozoderivatives (VI) of (IV). The products (VI), when warmed with hydrochloric acid, again yielded the amines (IV).

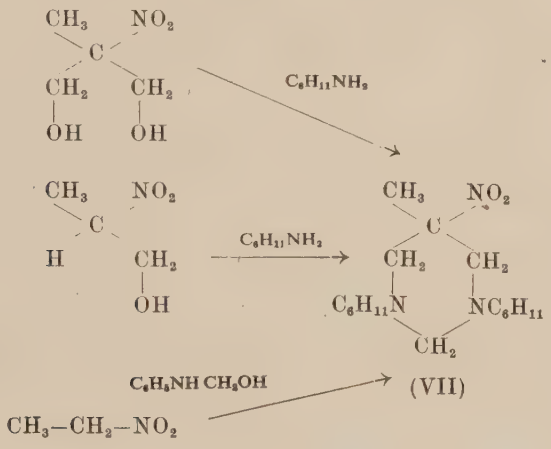
The products (IIb) and (IIc) were also prepared in a direct way on acting with cyclohexylamine on the corresponding diols: 2-ethyl-2-nitropropane-1,3-diol (i. e. $R = C_2H_5$) and 2-propyl-2-nitropropane-1,3-diol (i. e. $R = C_3H_7$), respectively, at room temperature.



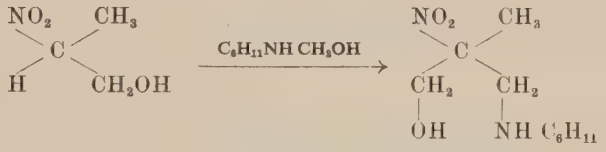
(II)

When $R = CH_3$, i. e. the derivative of nitroethane, was used, the reaction was very vigorous, a partial decomposition occurred, and a derivative of hexahydropyrimidine (VII) was formed. The dilution of reag-

ents with ether or alcohol did not alter the trend of the reaction. When the temperature was kept near 0°, the same product (VII) was obtained. The same product (VII) resulted, when the monohydric alcohol deriving from nitroethane (i. e. 2-nitro-propan-1-ol) reacted with cyclohexylamine at room temperature, or nitroethane reacted with N-hydroxymethyl-cyclohexylamine in the same conditions:

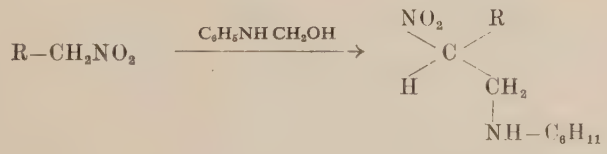


It was, however, found that (IIa), i. e. $R = CH_3$, could be obtained by acting on 2-nitropropan-1-ol with N-hydroxymethylcyclohexylamine:



(IIa)

The action of N-hydroxymethylcyclohexylamine on 1-nitropropane and 1-nitrobutane led to the formation, respectively, of the compounds (IVb, $R = C_2H_5$) and (IVc, $R = C_3H_7$):

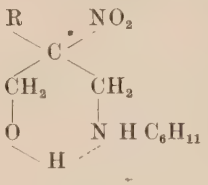


(IV)

Thus, the behaviour of nitroethane and its hydroxymethylderivatives differs from that of 1-nitropropane and 1-nitrobutane.

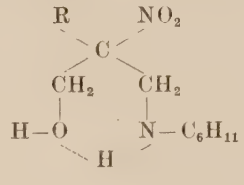
Appended is a list of the new compounds described in the present paper and the known compound (Ia), together with their melting points.

Attention is directed to the fact that the melting points of the oxazine derivatives (I) are lower than the m. pts. of the open chain aminoalcohols (II). This we ascribe to the existence of intramolecular hydrogen bonds, as previously suggested by one of us [3], [4]. In addition to the hydrogen bonds between the nitrogroup and hydrogen atoms of hydroxylic or aminogroups, hydrogen bonds between OH and NH C_6H_{11} groups can also be postulated, i. e. the structures (A) and (B):



(A)

or



(B)

TABLE

Name	Number	m. p.
5-nitro-5-methyl-3-cyclohexyl-tetrahydro-1,3-oxazine . . .	I a	50-51°
5-nitro-5-ethyl-3-cyclohexyl-tetrahydro-1,3-oxazine . . .	I b	59-60°
5-nitro-5-propyl-3-cyclohexyl-tetrahydro-1,3-oxazine . . .	I c	69-70°
I a hydrochloride		182° (d)
I b „		187° (d)
I c „		179° (d)
N-(-2-nitro-2-hydroxymethyl)-propyl-cyclohexylamine. . .	II a	68-70°
N-(-2-nitro-2-hydroxymethyl)-butyl-cyclohexylamine . . .	II b	77-79°
N-(-2-nitro-2-hydroxymethyl)-pentyl-cyclohexylamine. . .	II c	79-81°
II a hydrochloride		169° (d)
II b „		170° (d)
II c „		167° (d)
N-(2-nitro-propyl)-cyclohexylamine hydrochloride	IV a	148° (d)
N-(2-nitro-butyl)-cyclohexylamine hydrochloride	IV b	160° (d)
N-(2-nitro-pentyl)-cyclohexylamine hydrochloride	IV c	162° (d)
N-nitrosocompound derivatives of IV a.	VI a	61° (d)
„ „ „ „ IV b.	VI b	95° (d)
„ „ „ „ IV c.	VI c	97° (d)
N, O-dibenzoyl derivatives of II a	V a	106-107°
„ „ „ „ II b	V b	101-102°
„ „ „ „ II c	V c	140-142°

The experimental details will be reported elsewhere [5].

INSTITUTE OF ORGANIC CHEMISTRY, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] D. Gürne, T. Urbański, *On the synthesis and degradation of some derivatives of tetrahydro*-1,3-*oxazine*, Bull. Acad. Polon. Sci., Cl. III, **3** (1955), 175.

[2] M. Senkus, J. Am. Chem. Soc. **72** (1950), 2967.

[3] T. Urbański, *On the structure of some aliphatic nitrocompounds*, Bull. Acad. Polon. Sci., Cl. III, **1** (1953), 239.

[4] T. Urbański, D. Ciecierska, Roczniki Chem. **29** (1955), 11.

[5] D. Gürne, T. Urbański, Roczniki Chem. (in the press).

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

CHEMISTRY

On Acetylation of Bis-(2-hydroxymethyl-2-nitro)-butylamine

by

T. URBAŃSKI and B. SZCZYCIŃSKI

Communicated by T. URBAŃSKI at the meeting of November 28, 1955

It has been previously stated that the main product of the reaction between 1-nitropropane, formaldehyde and ammonia or 2-ethyl-2-nitropropane-1,3-diol, formaldehyde and ammonia is 5-ethyl-5-nitro-3-(2-hydroxymethyl-2-nitrobutyl)-tetrahydro-1,3-oxazine (I). On warming with hydrochloric acid (I) was subjected to hydrolysis with the loss of one mol. of formaldehyde, and bis-(2-hydroxymethyl-2-nitro)-butylamine (II) resulted [1].

In the present paper, experiments on acetylation of the product (II) are described.

When the hydrochloride of (II) was subjected at room temperature to the action of 3 mols of acetic anhydride in the presence of pyridine, a crystalline triacetyl derivative of (II) resulted with a yield of c. 80%.

When triacetyl derivative of (II) was subjected at 0°C to hydrolysis with aqueous solution of sodium hydroxide in presence of aceton (Kuntz method [2], [3]), no stepwise deacetylation was observed, and all three acetyl groups were hydrolysed in these experiments.

An attempt was made to prepare the N-acetyl derivative of (II) by acetylating the base (II) with acetic anhydride in an aqueous medium in the presence of hydrated sodium acetate. (It is well known, that this method can be applied to selectively acetylate aminogroups, without acetylating the hydroxylgroups [4]). However, instead of N-acetyl derivative of (II), the compound (I) resulted. The formation of (I) can be explained, in view of the partial decomposition of (II) accompanied by the evolution of formaldehyde.

According to the former experiments [1] the decomposition of (II), when boiled in water, led to the liberation of two mols of formaldehyde and the formation of 2-nitrobutene, 2-nitrobutanol and 5-nitro-5-ethyl-tetrahydro-1,3-oxazine.

The same results (the formation of (I) from (II) was obtained when acting with sodium acetate without acetic anhydride on (II).

The experiments described in the present paper constitute evidence of the particular ease with which the tetrahydro-1,3-oxazine ring is formed.

Experimental part

Acetylation of (II). 3.5 g. (0.011 mol.) of hydrochloride of (II) (m. p. 182-184°) were mixed at a temperature of 20°C. with 7.0 ml. (0.076 mol.) of acetic anhydride and 15 ml. of pyridine. The mixture was then left for 22 hours at room temperature. 20 ml. water were then added. A crystalline triacetyl derivative of (II) (yield — 3.6 g., i. e. 80% of theoretical) was precipitated. After crystallisation from ethanol it gave m. p. 68-69°.

Analysis:

found 46.7% C, 6.7% H, 10.6% N, 9.4% Ac;

$C_{10}H_{18}O_6N_3Ac_3$ requires 47.3% C, 6.7% H, 10.4% N, 9.35% Ac.

Action of acetic anhydride on (II) in an aqueous medium in the presence of sodium acetate. 2 g. of hydrochloride of (II) were dissolved in 14 ml. water, mixed with 0.7 g. acetic anhydride, and immediately after a solution of 0.8 g. sodium acetate hydrate in 3 ml. water was added. After warming on a steam-bath for a few minutes, the solution became cloudy and, on cooling, a crystalline product (I) precipitated (yield 0.95 g., i. e. c. 50% of theoretical). M. p. and mixed m. p. with the original sample of (I) 100-101°C.

Analysis:

found 14.55% N;

calculated for $C_{11}H_{21}O_6N_3$ — 14.4% N.

Action of aqueous sodium acetate on (II). 2 g. of hydrochloride of (II) were dissolved in 14 ml. water, mixed with 0.8 g. sodium acetate hydrate in 3 ml. water. After warming for a few minutes on a steam bath, the solution became cloudy and on cooling a crystalline precipitate of (I) resulted (yield 0.95 g.). M. p. and mixed m. p. with (I) 100-101°C.

INSTITUTE OF ORGANIC CHEMISTRY, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] E. L. Hirst, J. K. N. Jones, S. Minahan, F. W. Ochyński, A. T. Thomas, T. Urbański, J. Am. Chem. Soc. **69** (1947), 924.

[2] Kuntz, J. Am. Chem. Soc. **48** (1926), 1982.

[3] M. L. Wolfrom, M. Königsberg, S. Stoltzberg, J. Am. Chem. Soc. **58** (1936), 490.

[4] e. g. BIOS Final Report No. 1152, 88.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

CHEMISTRY

An Attempt to Elucidate the High Pressures of Hydrogen in Iron Cathodically Saturated with Hydrogen

by

K. GUMIŃSKI

Presented by M. ŚMIAŁOWSKI on January 17, 1956

M. Śmiałowski and Z. Szklarska-Śmiałowska [1], [2] have described a number of the phenomena occuring on iron cathodes on which hydrogen is being developed electrolytically. Closely connected with their work are the works of A. Krupkowski [3], W. Palczewska [4], [6], J. Foryst [5], W. Kozłowski [7], and H. Jarmołowicz [8].

From their research work it follows that in order to explain the facts observed, it must be assumed that very high pressures appear in connection with the penetration of the hydrogen into the metal. Krupkowski [3] estimates these pressures in certain cases as 10^7 atmospheres.

Although attempts to elucidate the origin of these pressures have been made by some of the authors mentioned, yet the question has not been definitively solved.

In this paper, an attempt is made to elucidate the appearance of such high pressures arising as a result of the penetration of hydrogen into iron.

I assume that the hydrogen enters the metal in the form of protons, and that these obtain the energy necessary to penetrate into the interior of the metal lattice in the strong electric field prevailing in the vicinity of the metal/solution contact (interface). I further assume that the protons entering the interior of the metal penetrate into the crystallographic lattice, and remain there as protons. Finally, I assume that the electrons corresponding to them belong to the common electronic gas of the metal.

Thus in these regions where the protons accumulate, there follows an increase in the density of the electronic gas. As a consequence, the pressure of this gas undergoes a change, in accordance with the statistical law of Fermi and Dirac.

From the publications cited, it follows that after 100-120 minutes of electrolysis, a certain state is established. In this state one can assume that the hydrogen content within the cathode (an iron wire) depends on the distance from the centre of the wire, probably as shown in the accompanying diagrams. Close to the surface of the wire, it is negligible; at a certain depth it rises and then again falls rapidly. When the radius is less than 0.15 cm, a certain quantity of hydrogen is found in the centre of the wire. With a radius greater than 0.15 cm, the hydrogen content in the centre amounts to zero. The accompanying diagrams show the probable dependence of the hydrogen content on the distance from the centre of the wire, when $r<0.15$, $r=0.15$, and $r>0.15$ cm.

Fig. 1

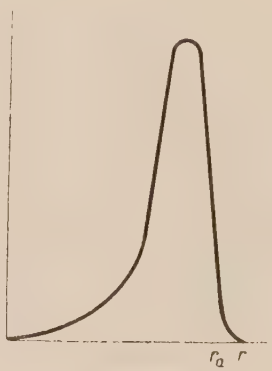


Fig. 2

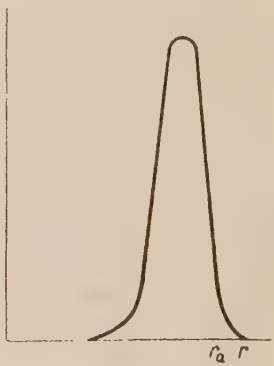


Fig. 3

From the course of these curves it follows that even when $r=0.15$ cm, the amount of hydrogen in the central part of the wire is small by comparison with the space corresponding to the maximum of the curve $\varrho(r)$. Practically, the hydrogen is accumulated in this active region. Krupkowski and Śmiałowski [3] introduce the notion of a radius r_a, denoting the radius defining the external boundaries of the layer, beyond which the presence of hydrogen may be disregarded. They give the following formula for the calculation of this radius:

$$r^a = 1 \cdot 1r - 0 \cdot 28 \quad \text{(in millimetres)}.$$

In accordance with the assumptions made above in this paper the pressure of the electronic gas would change in conformity with the course of the curves $\varrho(r)$. The gradient of this pressure would also change as a function of the distance from the centre of the wire, accepting the highest values in the region of the greatest rise and fall of the curves, and therefore on the boundaries of the active layer. Since the particular form and numerical values of the parameters of these curves are not yet known, I propose to give an approximate estimation of the values of these gradients. It will be accepted for this purpose that the hydrogen

content is a linear function of the distance from the centre of the wire and that hydrogen is found only in the region from $r=0$ to $r=r_a$. It will also be accepted that the content at the centre of the wire amounts to zero, which, according to what has already been said, is an acceptable assumption in this calculation. Since the field of the triangle thus created expresses the quantity of hydrogen absorbed, and must be equal to the field comprised by the actual curve of the hydrogen distribution, it is clear that the actual gradients corresponding to the active regions are greater — and may be considerably greater — than those obtained in such an approximate calculation. Since the region active in the phenomena appearing on the cathode corresponds to the maximum of the curve $\varrho(r)$, it seems probable that the values calculated will rather be the lower limit of the actual active quantities.

Consider the calculation for two cases. In the first, let the cathode be an iron wire of a radius $r=0.05$ cm. and a mass of 100 g., as in the experiments of Bardenhauer and Ploun, referred to by Śmiałowski and Szklarska-Śmiałowska. It absorbs 90.5 cc. of hydrogen, and so 0.004 mols.

In the second case, let the cathode be an iron wire of a radius of 0.15 cm., as investigated by Śmiałowski and Krupkowski [3]. From Bardenhauer and Ploun's curves [9], it may be read that for such a wire the amount of hydrogen absorbed by 100 g. Fe is about 62 cc., and so about 0.0027 mol.

Denoting by x the absorbed number of mols of protons, the length of the immersed cathode by l, its radius by r, the proton density in mols per cc. by ϱ, the coefficient of the linear function $\varrho(r)$ by a, the mass of the cathode by M, and its density by d, we obtain:

$$x=2\pi l\int_0^{r_a} r\varrho\,dr,$$

$$\varrho=ar,$$

whence

$$a=\frac{3}{2}\,\frac{d}{M}\,\frac{r^2}{r_a^3}\,x$$

where, of course,

$$l=\frac{M}{r^2\pi d}.$$

For the temperature of absolute zero, in the case of extreme degeneration, the pressure of electronic gas is given by the expression:

$$p=\frac{5}{2}\left(\frac{3}{8\pi}\right)^{2/3}\frac{h^2}{2m}\left(\frac{n}{V}\right)^{5/3}z^{5/3}$$

or, more briefly,

$$p=p_0z^{5/3}.$$

In these equations,

m denotes the electron mass,
V „ the volume of metal,
h „ Planck's constant,
n „ the number of metal atoms in the volume V,
z „ the number of electrons which one atom of metal yields to the conductance band.

Substituting particular values, we find for p_0 the value

$$p_0 \approx 4.10^5 \text{ atm.}$$

From the last equation it follows that:

$$\frac{dp}{dr} = \frac{5}{3} p_0 z^{2/3} \cdot \frac{dz}{dr}.$$

Treating formally the electrons, the presence of which in the conductance band is caused by the penetration of the protons into the metal lattice, as an increase in the number z, we can write

$$z = z_0 + \lambda,$$

where z_0 denotes the number of electrons which the metal atom itself yielded to the conductance band and λ denotes the number of electrons evoked by the presence of the protons and calculated as an average per one metal atom in the given region. In accordance with this definition,

$$\lambda = \frac{\mathcal{M}}{d} \varrho = \frac{\mathcal{M}}{d} \cdot a \cdot r,$$

where $\mathcal{M}$ denotes the atomic mass of the metal. Hence

$$\frac{d\lambda}{dr} = \frac{\mathcal{M}}{d} a.$$

Since

$$\frac{dz}{dr} = \frac{d\lambda}{dr},$$

therefore

$$\frac{dp}{dr} = \frac{5}{3} p_0 z^{2/3} \frac{\mathcal{M}}{d} \cdot a$$

and finally

$$\frac{dp}{dr} = \frac{5}{2} p_0 z^{2/3} \frac{\mathcal{M}}{M} \frac{r^2}{r_a^3} r.$$

Substituting

$$p_0 = 4.10^5,$$
$$z = 6,$$
$$\mathcal{M} = 56,$$

$$M = 100,$$
$$r = 0.05,$$
$$r_a = 0.027,$$
$$x = 0.004,$$

we find

$$\frac{dp}{dr} \approx 9 \cdot 3.10^5 \frac{\text{atm.}}{\text{cm.}} \approx 10^6 \frac{\text{atm.}}{\text{cm.}}.$$

In the second case, by substituting

$$r = 0.15,$$
$$r_a = 0.137,$$
$$x = 0.0027,$$

we find

$$\frac{dp}{dr} \approx 0 \cdot 4.10^5 \frac{\text{atm.}}{\text{cm.}}.$$

We thus obtain in both cases high gradients of pressures for $T = 0$. For a temperature of 300°K, the following formula should be applied:

$$p = p_0 z^{5/3} + \frac{1}{5} \cdot \frac{1}{p_0} \left(\frac{n}{V}\right)^2 \frac{\pi^2 k^2}{3} z^{1/3} T^2.$$

As may easily be verified by differentiating the above expression in relation to r and substituting the numerical values, the temperature has no significant influence on the gradient of pressure.

The gradients obtained are, as has been mentioned, certainly considerably smaller than those appearing on the steep parts of the actual curve in the active region. It should be remembered that the calculation was made for the stage when a part of the hydrogen is no longer in a protonic state. Hence the value accepted for x is too high. On the other hand, however, it should be taken into account that before the establishment of this state, the entire quantity of hydrogen absorbed has passed through the protonic state. The active layer in the process of formation also corresponded in the first stages of the process to a much steeper and narrower curve. This curve in the course of time has been "dispersed" by diffusion, becoming increasingly slender, and finally taken its definitive shape. Hence it follows that the gradients actually active will be considerably greater at the beginning of the process and will diminish as it approaches the final state. The same will happen, if the final state would be, as a result of diffusion, a uniform partition of hydrogen in the cathode. From all these considerations, it may be assumed that the values calculated approximate rather to the lower order of magnitudes appearing during the process.

It seems to be possible, therefore, to explain the phenomena occurring on the iron cathode during the electrolytical development of hydrogen in the following way. As a result of the process of electrolysis, protons

penetrate the iron cathode. This causes a disturbance in the state of the metal shown by, among other things, considerable differences in the energy of the electronic gas. These differences are expressed by the great gradients in the pressure of the electronic gas. This pressure is connected with the energy of the electrons by the fundamental formula:

$$p = \frac{2}{3} \cdot \frac{E}{V}.$$

As has already been said, these differences in the energy of the electronic gas and its pressure gradients constitute evidence of the disturbance in the state of the metal as a result of the introduction of protons into the metal lattice, and may be to a certain extent a measure of the energetic differences in the latter arising in various regions of the metal. It may be assumed that this state of nonequilibrium in the metal evokes a tendency to give off protons from the network to the space in which they could pass into the atomic and then the molecular state. This would clarify the deformations of the metal lattice occurring in the weaker places (especially at lattice defects), the probable lattice deformation, the formation of cleavages between the crystals, and the augmentation of these cleavages and of the already existing micropores as Krupkowski supposes. It may be assumed that, between the hydrogen accumulating in the closed clevages and micropores and the protonic hydrogen in the metal lattice, an equilibrium would be established. On account of the fact that, as follows from the above considerations, the giving off of protons from the metal lattice is connected with the liberation of enegry, the pressure of the hydrogen accumulating in the cleavages and micropores may be very high. The hydrogen accumulating in these cleavages and micropores might therefore cause further mechanical consequences as assumed by Krupkowski and Śmiałowski [3].

DEPARTMENT OF THEORETICAL CHEMISTRY, JAGELLONIAN UNIVERSITY, CRACOW
LABORATORY OF PHYSICOCHEMISTRY OF ELECTRODE PROCESSES, INSTITUTE OF PHYSICAL CHEMISTRY, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] M. Śmiałowski and Z. Szklarska-Śmiałowska, Bull. Acad. Polon. Sci., Cl. III, **1** (1953), 159; **1** (1953), 237; **2** (1954), 73, Roczn. Chemii **27** (1954), 505; **29** (1955), 85.

[2] M. Śmiałowski and Z. Szklarska-Śmiałowska, Proceedings of the Electrochemical Conference, Warsaw, 1955.

[3] A. Krupkowski and M. Śmiałowski, l. c.

[4] W. Palczewska and M. Śmiałowski l. c.

[5] J. Foryst and M. Śmiałowski l. c.

[6] W. Palczewska l. c.

[7] L. Kozłowski l. c.

[8] H. Jarmołowicz l. c.

[9] M. Śmiałowski, *Wodór w stali*, 1947.

BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Cl. III — Vol. IV, No. 4, 1956

CHEMISTRY

The Critical Region of Liquids. I. Liquid-Vapour Miscibility and Retrograde Condensation

by

A. KRĘGLEWSKI

Presented by W. ŚWIĘTOSŁAWSKI on February 3, 1956

Introductory remarks

A review of the literature shows that a large number of investigations on the critical state of liquids have been accumulated, especially in the two last decades. Recent works on the influence of gravity on critical phenomena [1]-[3] have shown that, if a necessary correction for this gravity effect is introduced, the shape of the isotherms and the liquid vapour coexistence curve is adequately described by the continuity theory. The phenomena observed above the critical temperature T_m of pure liquids, especially those associated with the hysteresis, will be discussed in another paper of the present series [6].

The purpose of this paper is to show, on the basis of the influence of gravity and diffusion phenomena in the critical region of pure liquids, that the so-called retrograde condensation is a result of the liquid-vapour miscibility in the mixtures.

The properties of the liquidosol

The phenomenon of mutual liquid-vapour solubility was first observed by Traube [9] who gave the modern definition of the critical temperature T_m as "that temperature at which the two solubility curves intersect, and in which the liquid and the gas phases are mutually soluble in all ratios". It has been shown further by Mason, Naldrett and Maass [10], [11] that at T_m the opalescence reaches a sharp maximum and, hence, it is identical with the critical temperature as defined by Smoluchowski [10], [11].

The terms *liquidosol* and *gaseosol* were introduced by Świętosławski [7] with a view to explaining the results of investigations carried out by the dynamic method which he was the first to describe and use. The

liquidosol is a solution of vapour in the liquid phase. The gaseosol is a solution of clusters in the gaseous phase. The term *cluster* is used here (as in previous papers by other authors quoted in [17]) to emphasize the transition of colloidal particles of different sizes into the density fluctuations. The temperature at which the transition of the liquidosol into the gaseosol was observed by the dynamic method [7] seems to be close to the temperature T_m [12]. Consequently, the "vapour" coexisting with the liquidosol possesses practically the same nature as the gaseosol above T_m.

If a liquid-vapour system is heated at constant volume at a temperature close to T_m, different phenomena are observed depending upon the mean density $\bar{\varrho}$ of the system (Fig. 1).

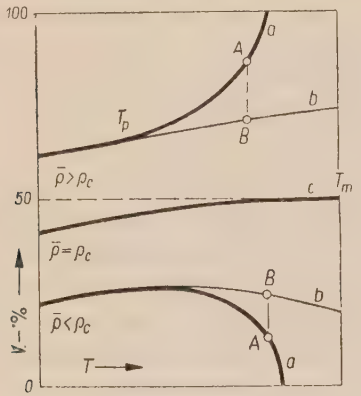


Fig. 1.

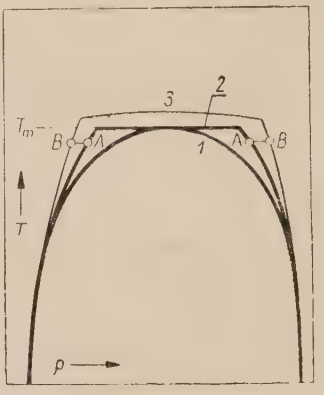


Fig. 2.

Fig. 1. Changes in the meniscus level (expressed in vol. per cent of the liquid phase) as function of the temperature and mean density $\bar{\varrho}$ of the system.

Fig. 2. Liquid-vapour coexistence curve for pure substances. Curves 1 and 2 are the equilibrium curves according to Weinberger and Schneider [1]. Curve 3 corresponds to a metastable state.

If $\varrho > \varrho_c$ (ϱ_c being the critical density), and the system is heated without stirring, the meniscus level changes along the curve *b*. Since the two phases mix together very slowly, the curves *b* (Fig. 1) and 3 (Fig. 2) represent at the first moment the amounts and densities of the almost "pure" liquid and vapour phases. In this case the meniscus disappears at a temperature somewhat higher than T_m. Above a certain temperature T_p, the turning of the tube produces a formation of both the liquido- and the gaseosols. This is accompanied by the displacements of the meniscus level and the changes in the density of the two phases [13] (*A* on Fig. 1 and 2). A decrease of pressure of the order of 0.01 atm, associated with this process [4]-[5], and an increase in opalescence [10], [11] were observed. The heat effect of these phenomena is, if any, very small [4]-[5].

On cooling, the system transforms itself along curve *a*, and a violent evolution of vapour from the liquidosol is observed.

If $\bar{\varrho} < \varrho_c$, opposite phenomena are observed. The volume of the liquidosol decreases along curve *a*, until the dew point is reached.

If $\bar{\varrho} = \varrho_c$, the mixing produces no visible change. Therefore the volume of vapour dissolved in the lower phase is approximately equal to that of liquid dissolved in gaseosol.

The diffusion region, characterised by transition from curves 3 to 2 (Fig. 2), is associated with metastable states. Curve 2 corresponds to the *equilibrium* coexistence of *saturated* liquidosol and gaseosol. According to this, in a certain range of $\bar{\varrho}$, the meniscus disappears at constant temperature equal to T_m [10], [11]. It has been shown, however, by Weinberger and Schneider [1], that, due to the gravitational effect, the shape of the equilibrium coexistence curve depends upon the vertical length of the tube. As the tube filled with a fluid is shortened the horizontal section of curve 2 decreases and transforms itself into curve 1, characteristic for the "zero length" of the tube.

Retrograde condensation

The retrograde condensation clearly observed in the critical region of mixtures was formally explained by Kuenen [14] on the basis of the assumption that the continuity theory is valid also for mixtures. According to Katz and Kurata [15] the "phase behaviour may be adequately described without using the term retrograde condensation". The existence of this interesting phenomenon cannot, however, be questioned, and the reasons of its appearance are unknown.

Below is outlined a simple explanation of the retrograde phenomena. It arises from a direct comparison of critical phenomena known in the case of pure substances, with those observed in mixtures.

Andrews and several other investigators [14] compressed certain mixtures isothermally at temperatures lying above the critical temperature. These compressions were carried out *without stirring*. The investigators found that the amount of the liquid phase increased steadily, and finally the meniscus disappeared at a certain level of the tube. No retrograde condensation was observed. Kuenen [14] repeated these experiments *with stirring* and observed in all cases retrograde condensation of the first kind, i. e., a fall in the liquid level from a certain point until a dew point was reached. He believed this discrepancy to be due to "Verzögerungserscheinungen" in the absence of stirring, and a consequence of the influence of gravity.

It should be emphasized that, in the critical region of mixtures, the gravity effect is very small, because the compressibility is not infinite as in the case of pure substances. Therefore in tubes of different lengths, only one *equilibrium* coexistence curve may be obtained for a given mixture.

It is well known that the changes in the meniscus levels are identical in the case of mixtures and of pure substances (Fig. 1). Generally speaking, the solvent power of vapour in relation to its own liquid increases with the density of the vapour. This causes a further increase in gaseosol density, and, in the words of Scott [16], "an acceleration in the disordering process sets in". The density of vapour can, in pure substances, be increased only by heating (at constant volume), and in mixtures — by heating or by isothermal compression. Therefore the experimental results referred to above [14] may be directly compared with those shown in Fig. 1 and the following conclusions formulated: (a) the retrograde condensation of the first kind is a result of the dissolution of the liquid in the vapour phase, and it may take place only if $\bar{\varrho} < \varrho_c$; (b) the retrograde condensation of the second kind (or retrograde vaporisation) is a result of dissolution of the vapour in the liquid, and therefore it may appear only, if $\bar{\varrho} > \varrho_c$.

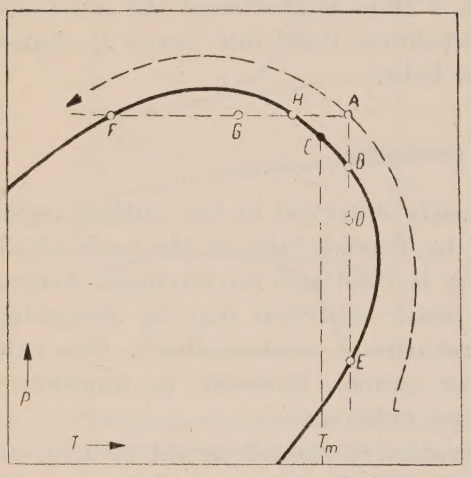


Fig. 3. Pressure-temperature diagram in the critical region of a mixture.

It seems reasonable to describe the critical phenomena of a mixture by using the $P-T$ diagram, which further clarifies these phenomena.

If we proceed along line L (Fig. 3), the system undergoes a gradual change from a gas, through the gaseosol and the liquidosol (below T_m) and reaching finally the liquid state. Suppose we expand the system isothermally from point A. At point B, the first droplet of the liquidosol appears, being in equilibrium with the saturated gaseosol. In the course of a further expansion, the amount of "pure" liquid *decreases* (as in the one component system); the solubility of the liquid in the vapour however decreases more rapidly. Consequently, the amount of the liquidosol *phase* increases. At point D the amount of the more dense phase reaches a maximum.

At this point, the *two oppositely directed phenomena — vaporisation and liquid — vapour miscibility* — are balanced one with the other. From point D, vaporisation becomes predominant and the amount of the liquid decreases to zero at point E.

If the system is cooled down from point A, with such compression as to keep the pressure constant, point H will be reached, at which the first bubbles of saturated gaseosol appear. On further cooling the decrease of solubility of vapour in the liquidosol is so rapid that it cannot be compensated by the increase of the amount of "pure" liquid resulting

from the compression. Consequently, the meniscus drops. At point G, a minimum is reached, and on further continuation of the process, the vapour undergoes condensation until the bubble point F.

At point C are observed typical critical phenomena. The meniscus disappears in the middle of the tube at T_m.

The above considerations warrant the conclusion that there are no essential differences in the critical phenomena observed in pure substances and in mixtures.

I wish to express my thanks to Professor W. Świętosławski for many helpful suggestions and comments.

Summary

It becomes clear, on the basis of a discussion of gravity and diffusion phenomena, well known in the case of pure substances, that what is called retrograde condensation, a characteristic phenomenon in the critical region of mixtures, is a result of the solubility changes of the liquid phase in gaseous phase and vice versa. The position of the meniscus level depends on two oppositely directed phenomena: vaporisation (or condensation) and liquid-vapour miscibility.

INSTITUTE OF PHYSICAL CHEMISTRY, POLISH ACADEMY OF SCIENCES

REFERENCES

[1] M. A. Weinberger, W. G. Schneider, Can. J. Chem. **30** (1952), 422.

[2] H. W. Habgood, W. G. Schneider, Can. J. Chem. **32** (1954), 98; J. Chem. Phys. **21** (1953), 2080.

[3] H. D. Baehr, Z. Elektrochem. **58** (1954), 416.

[4] O. Maass, Chem. Revs. **23** (1938), 17.

[5] D. B. Pall, J. W. Broughton, O. Maass, Can. J. Research **B-16** (1938), 230.

[6] A. Kręglewski, Bull. Acad. Polon. Sci. Cl. III (1956) printing.

[7] W. Świętosławski, Bull. Acad. Polon. Sci. **1938-A**, 304; *Ebulliometric Measurements*, p. 186, Reinhold Publ. Corp., New York, 1945.

[9] I. Traube, Z. physik. Chem. **58** (1907), 475; Trans. Faraday Soc. **34** (1938), 1234.

[10] S. G. Mason, S. N. Naldrett, O. Maass, Can. J. Research **B-18** (1940), 103.

[11] S. G. Mason, O. Maass, ibid. **B-26** (1948), 592.

[12] A. Kręglewski, Roczniki Chem. **29** (1955), 95.

[13] W. Świętosławski, A. Kręglewski, Roczniki Chem. **26** (1952), 433.

[14] J. P. Kuenen, *Theorie der Verdampfung u. Verflüssigung von Gemischen*, Leipzig 1906.

[15] D. L. Katz, F. Kurata, Ind. Eng. Chem. **32** (1940), 817.

[16] R. L. Scott, J. Chem. Phys. **21** (1953), 209.

[17] A. Kręglewski, Bull. Acad. Polon. Sci. Cl. III, **2** (1954), 233.